General Stuff

- Office Hours

  T: 12:30 - 1:30, Th: 10 - 11
- Lab 3 due tonight

  Exercises 1,2,5
- Midterm Thursday 2/18

  2 problems

  30 minutes to take exam

  5-10 minutes to upload to gradescope

  11:15 - 11:25 questions before ~~quiz~~ midterm

  11:25 - 11:55 midterm

  11:55 - 12:05 uploading
- Lab after midterm from 12:20 - 1:10

2.4 - 2.6

Chain rule

directional derivative

$v_1 = \left(1, 0, \frac{\partial f}{\partial x}\right)$

$v_2 = \left(0, 1, \frac{\partial f}{\partial y}\right)$

$n = v_1 \times v_2$

$= \nabla \left(z - f(x,y)\right)$

1. Let $f(x, y) = x^2 + y^2$ and $p(t) = (3\cos(t) - 1, 3\sin(t) + 2)$. Find $\frac{d}{dt}(f \circ p)$.

$p : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ parametrization of a curve

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$ 2D scalar function

$(f \circ p)(t) = f(p(t)) \; : \; \boxed{\mathbb{R}^1} \xrightarrow{p} \mathbb{R}^2 \xrightarrow{f} \boxed{\mathbb{R}^1}$

$f \circ p : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$$(f \circ p)(t) = f(p(t)) = f(3\cos(t) - 1, 3\sin(t) + 2)$$

$$= (3\cos(t) - 1)^2 + (3\sin(t) + 2)^2$$

$$\frac{d}{dt}(f \circ g) = \frac{d}{dt}\left((3\cos(t) - 1)^2 + (3\cos(t) + 2)^2\right)$$

let's do it the multi way!

$$\overset{\boxed{1 \times 1}}{D((f \circ p)(t))} = \underbrace{\overset{\boxed{1 \times 2}}{Df(p(t))}}\,\underbrace{\overset{\boxed{2 \times 1}}{Dp(t)}}$$

$$p(t) = (3\cos(t) - 1,\ 3\sin(t) + 2) \qquad f(x,y) = x^2 + y^2$$

$$Dp(t) = \begin{array}{c} t \\ \begin{bmatrix} -3\sin(t) \\ 3\cos(t) \end{bmatrix} \end{array}$$

$$Df(x,y) = \begin{array}{cc} x & y \\ \end{array}$$
$$Df(x,y) = \begin{bmatrix} 2x & 2y \end{bmatrix}$$

$$x = 3\cos(t) - 1$$
$$y = 3\sin(t) + 2$$

$$Df(p(t)) = \begin{bmatrix} 2(3\cos(t) - 1) & 2(3\sin(t) + 2) \end{bmatrix}$$

$$\frac{d}{dt}\left((f \circ p)(t)\right) = \begin{bmatrix} 6\cos(t) - 2 & 6\sin(t) + 4 \end{bmatrix} \begin{bmatrix} -3\sin(t) \\ 3\cos(t) \end{bmatrix}$$

$$= \cancel{-18\cos(t)\sin(t)} + 6\sin(t) + \cancel{18\sin(t)\cos(t)} + 12\cos(t)$$

$$= 6\sin(t) + 12\cos(t)$$

$p : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^1$ $\quad f(x, y, z)$ $\quad$ these functions don't compose

$f \circ p : \mathbb{R}^1 \xrightarrow{p} \mathbb{R}^{2}$ 3
$\quad \mathbb{R}^3 \xrightarrow[f]{} \mathbb{R}^1$

$f \circ p = f(p(t))$ makes no sense

$p(t) = 3\cos t - 1, \; 3\sin(t) + 2 \qquad z = ??$

$f \circ p : \mathbb{R}^{1} \xrightarrow{p} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^{1}$

$D(f \circ p) \quad 1 \times 1 \quad =$

$$= Df(p(t)) \, Dp(t) = \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \end{bmatrix}$$

2. Let $F(x,y) = (x^2, y^2, x+y, xy+e^x)$ and $G(s,t,u,v) = (st, uv, 3t^2)$. Find $D(G\circ F)(0,2)$.

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ $\quad DF \quad 4\times 2$

$x, y \qquad x^2, y^2, x+y, xy+e^x$

$G : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ $\quad DG \quad 3\times 4$

$s, t, u, v \qquad st, uv, 3t^2$

$G\circ F : \mathbb{R}^2 \xrightarrow{F} \mathbb{R}^4 \xrightarrow{G} \mathbb{R}^3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$D(G\circ F) \quad 3\times 2$

$G(F)$

$$(G\circ F)(x,y) = G(\overset{s}{x^2}, \overset{t}{y^2}, \overset{u}{x+y}, \overset{v}{xy+e^x}) = D(G)(F(x))\, DF(x)$$

$$3\times 4 \quad 4\times 2$$

$$= \left(x^2y^2,\ (x+y)(xy+e^x),\ 3(y^2)^2\right)?$$

$$D(G\circ F)(0,2) = \begin{bmatrix} 2xy^2 & 2x^2y \\ & \end{bmatrix}$$

we could just find the total deriv of this and get the answer!

$$D(G\circ F)(0,2) = DG(F(0,2))\,DF(0,2)$$

$\boxed{3\times 2}$ $\quad$ 3×4 $\quad$ 4×2

$$F(x,y) = (x^2, y^2, x+y, xy+e^x)$$

$$DF = \begin{array}{c} \\ x^2 \\ y^2 \\ x+y \\ xy+e^x \end{array} \begin{array}{c} x \qquad y \\ \begin{bmatrix} 2x & 0 \\ 0 & 2y \\ 1 & 1 \\ y+e^x & x \end{bmatrix} \end{array}$$

$$DF(0,2) = \begin{bmatrix} 0 & 0 \\ 0 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$$

4×2

$$G(s,t,u,v) = (st, uv, 3t^2)$$

$$DG = \begin{array}{c} \\ st \\ uv \\ 3t^2 \end{array} \begin{array}{c} s \quad t \quad u \quad v \\ \begin{bmatrix} t & s & 0 & 0 \\ 0 & 0 & v & u \\ 0 & 6t & 0 & 0 \end{bmatrix} \end{array}$$

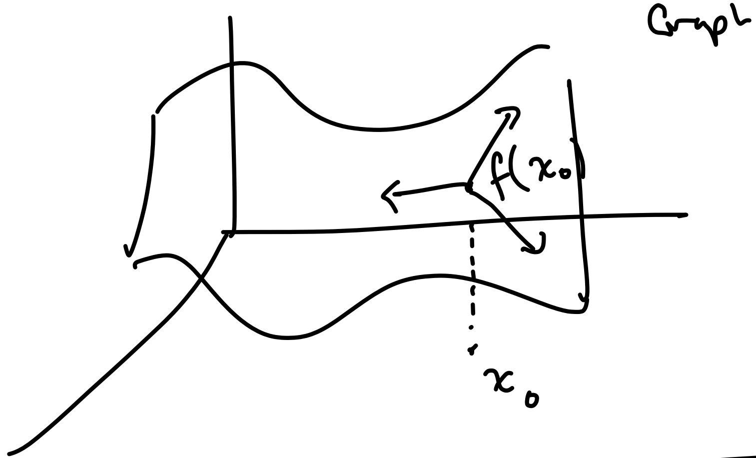
$$\begin{array}{c} DG(0,4,2,1) \\ DG(F(0,2)) \end{array} = \begin{array}{c} s \quad t \quad u \quad v \\ \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 24 & 0 & 0 \end{bmatrix} \end{array}$$

$$F(0,2) = (0^2, 2^2, 0+2, 0\cdot 2+e^0) = (0,4,2,1)$$

$$D(G\circ F)(0,2) = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 24 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 7 & 1 \\ 0 & 96 \end{bmatrix} = \begin{array}{c} x \qquad\qquad y \\ \begin{bmatrix} \frac{\partial(G\circ F)}{\partial x} & \frac{\partial(G\circ F)}{\partial y} \\ e+2 & \end{bmatrix} \end{array}$$

3. Let $f(x, y, z) = xyz + xy + xz + yz + 2$. Find the directional derivative of f in the direction $w = (2, -1, 1)$ at $\boxed{x_0 = (0, 3, 1)}$.

$f(x,y)$



$\nabla_{\vec{u}} f =$ "slope of the graph of f if you were to walk in the direction $\vec{u}$" intuition

$= \nabla f \cdot \vec{u}$ formula

$\vec{u}$ is a $\boxed{\text{unit vector}}$! $\|\vec{u}\| = 1$

$f(x,y,z) = xyz + xy + xz + yz + 2$

$w = (2,-1,1)$ direction

NOT a unit vector

$$\vec{u} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{(2,-1,1)}{\sqrt{2^2 + (-1)^2 + 1^2}}$$

$$= \frac{1}{\sqrt{6}}(2,-1,1)$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\nabla_u f(x_0) = \nabla f(x_0) \cdot \vec{u} = \underbrace{\nabla f(x_0)} \cdot \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$= \left(yz + y + z, \quad xz + x + z, \quad xy + x + y \right)$$

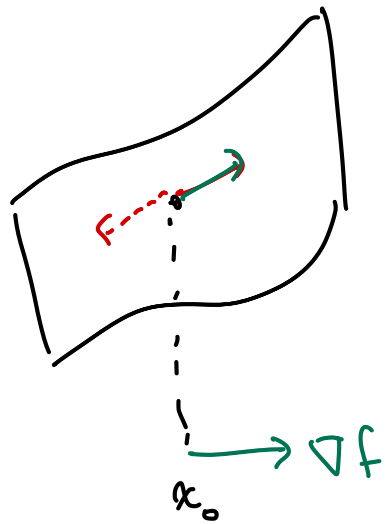
$$\nabla f(0,3,1) = \left(3 \cdot 1 + 3 + 1, \quad 0 \cdot 1 + 0 + 1, \quad 0 \cdot 3 + 0 + 3 \right)$$

$$= \underbrace{(7, 1, 3)}$$

$$\nabla_{(2,-1,1)\frac{1}{\sqrt{6}}} f(0,3,1) = (7,1,3) \cdot \left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

$$= \frac{14 - 1 + 3}{\sqrt{6}} = \frac{16}{\sqrt{6}}$$ = change in f in direction $(2,-1,1)$

- 4. Let $p(t) = (3t + e^t, -t^2)$ and $F(x, y) = (x^2, xy)$. Find $D(F \circ p)(0)$ using the chain rule.
- 5. Suppose the temperature at a point (x, y, z) in a room the shape of $\mathbb{R}^3$ is given by $T(x, y, z) = e^{-(x^2+y^2+z^2)}$. Furthermore, you are standing at the point $(x_0, y_0, z_0) = (1, -1, 0)$. What direction do you walk in order to have the maximum increase in temperature?



Graph of f

Which direction is the steepest slope on the graph of f?

What's the maximal directional derivative?

Which $\vec{u}$ makes $\nabla_{\vec{u}} f$ the biggest

Ans: $\vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$ maximizes $\nabla_u f$

4. Let $p(t) = (3t + e^t, -t^2)$ and $F(x, y) = (x^2, xy)$. Find $D(F \circ p)(0)$ using the chain rule.

$p : \mathbb{R}^1 \longrightarrow \mathbb{R}^2$ $\quad Dp \quad 2\times 1$

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad DF \quad 2\times 2$

$F \circ p : \mathbb{R}^1 \xrightarrow{p} \mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2 \implies \mathbb{R}^1 \xrightarrow{F \circ p} \mathbb{R}^2$ $\quad \boxed{2\times 1} = 2\times \boxed{2 \cdot 2} \times 1$

$$D(F \circ p)(0) = DF(p(0))\, Dp(0)$$

$$Dp = \begin{bmatrix} 3 + e^t \\ -2t \end{bmatrix} \qquad Dp(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \qquad p(0) = (1, 0)$$

$$DF = \begin{bmatrix} 2x & 0 \\ y & x \end{bmatrix} \qquad DF(p(0)) = DF(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D(F \circ p)(0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \checkmark$$

2×1 like we wanted

5. Suppose the temperature at a point (x, y, z) in a room the shape of $\mathbb{R}^3$ is given by $T(x, y, z) = e^{-(x^2+y^2+z^2)}$. Furthermore, you are standing at the point $(x_0, y_0, z_0) = (1, -1, 0)$. What direction do you walk in order to have the maximum increase in temperature?

Maximize change in temperature over $\vec{u}$ $= \max \{ \nabla_{\vec{u}} T \mid \vec{u} \text{ unit vectors} \}$

Ans: $\vec{u} = \dfrac{\nabla f}{\|\nabla f\|}$ maximizes $\nabla_u f$

So $\vec{u} = \dfrac{\nabla T}{\|\nabla T\|}$ is that direction! $(x_0 = (1,-1,0))$

$T = e^{-(x^2+y^2+z^2)}$

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) = \left(-2x\, e^{-(x^2+y^2+z^2)}, -2y\, e^{-x^2-y^2-z^2}, -2z\, e^{-x^2-y^2-z^2} \right)$$

Plug in $(1,-1,0)$

$$\nabla T(1,-1,0) = \left(-2e^{-2}, 2e^{-2}, 0 \right) = e^{-2}(-2, 2, 0)$$

5. Suppose the temperature at a point (x, y, z) in a room the shape of $\mathbb{R}^3$ is given by $T(x, y, z) = e^{-(x^2+y^2+z^2)}$. Furthermore, you are standing at the point $(x_0, y_0, z_0) = (1, -1, 0)$. What direction do you walk in order to have the maximum increase in temperature?

$v = (2, -2, 0)$ and $e^{-2}(2, -2, 0)$ have the same unit vector!!

$$\vec{u} = \frac{\nabla T}{\|\nabla T\|} = \frac{(2,-2,0)}{\sqrt{8}} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right) \quad \square$$