## General Stuff

- Office Hours

  T: 12:30 - 1:30, Th: 10 - 11

- Midterm Thursday 2/4

  2 problems

  30 minutes to take exam

  5-10 minutes to upload to gradescope

  ~~12:20 - 12:30~~ questions before midterms 11:15 - 11:25

  ~~12:30 - 1:00~~ midterm 11:25 - 11:55

  ~~1:00 - 1:10~~ uploading 11:55 - 12:05

- Lab 1 due tonight

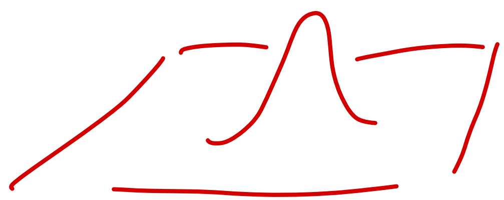
  Hand in Exercises 1,3, and 6

  For problem 1, match all 6 functions to their plots, recreate each plot best you can, and only describe plot 4 in terms of its function

Topics: 1.3, 1.5, (1.2 ... dot product magnitude)

Same policies
- camera during test
- no book / no calculator

Files → Labs → lab01.nb
→ graphs1b.pdf



1. Let

$$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} -6 & 1 \\ 4 & 0 \end{pmatrix} \quad v = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}.$$

Find which combinations of $A$, $B$, and $v$ can be multiplied and evaluate them.

$A = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix}$ $\qquad B = \begin{pmatrix} -6 & 1 \\ 4 & 0 \end{pmatrix}$ $\qquad \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \longleftrightarrow (-1, 1, 3)$

$3 \times 2$ (rows, columns) $\qquad 2 \times 2$ (r, c) $\qquad 3 \times 1$ (r, c) column vector

$AB$ multiply when $\underset{A}{n \times \boxed{m}}$ and $\underset{B}{\boxed{m} \times p}$

✓ $AB$ $\quad 3 \times \boxed{2 \quad 2} \times 2$ $\qquad \neq \qquad$ $BA$ $\quad 2 \times \boxed{2 \quad 3} \times 2$ ✗

$Av$ $\quad 3 \times \boxed{2 \quad 3} \times 1$ ✗ $\qquad vB$ $\quad 3 \times \boxed{1 \quad 2} \times 2$ ✗

$Bv$ $\quad 2 \times \boxed{2 \quad 3} \times 1$ ✗ $\qquad vA$ $\quad 3 \times \boxed{1 \quad 3} \times 2$ ✗

$$AB = \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -6 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} \boxed{x} & \boxed{x} \\ x & x \\ x & x \end{pmatrix}$$

$$\boxed{3} \times \boxed{2 \quad 2} \times \boxed{2} \qquad 3 \times 2$$

$$= \begin{pmatrix} 2\cdot(-6) + (-1)\cdot 4 & 2\cdot 1 + (-1)\cdot 0 \\ 12 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -16 & 2 \\ 12 & 0 \\ 2 & 1 \end{pmatrix}$$

Chain rule in multi is matrix multiplication ...

$$v = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \qquad v^T = \text{"}v \text{ transpose"} = \begin{pmatrix} -1 & 1 & 3 \end{pmatrix}$$

$$v^T A \qquad \boxed{1} \times \boxed{3 \quad 3} \times \boxed{2} \qquad \begin{pmatrix} -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 10 \end{pmatrix}$$

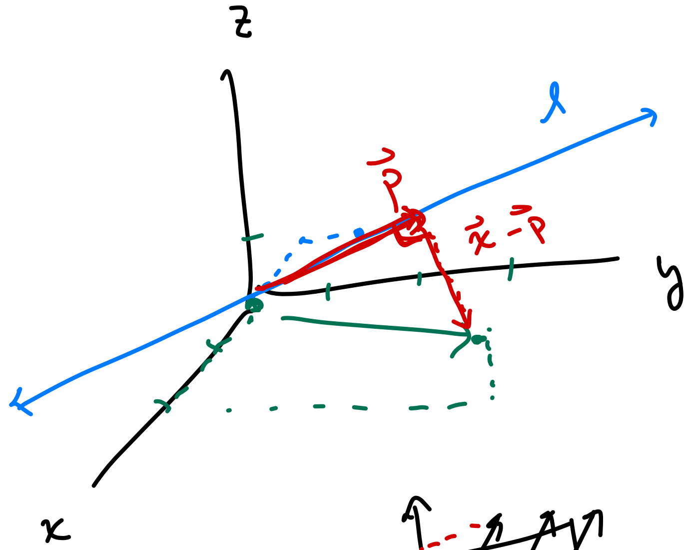
$$1 \times 2$$

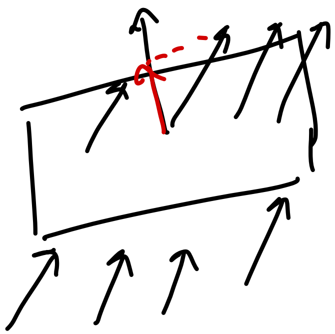
2. Project the vector $\underbrace{(2,3,1)}_{x}$ onto the line $\ell(t) = t\overbrace{(-1,1,0)}^{v}$ using the projection formula

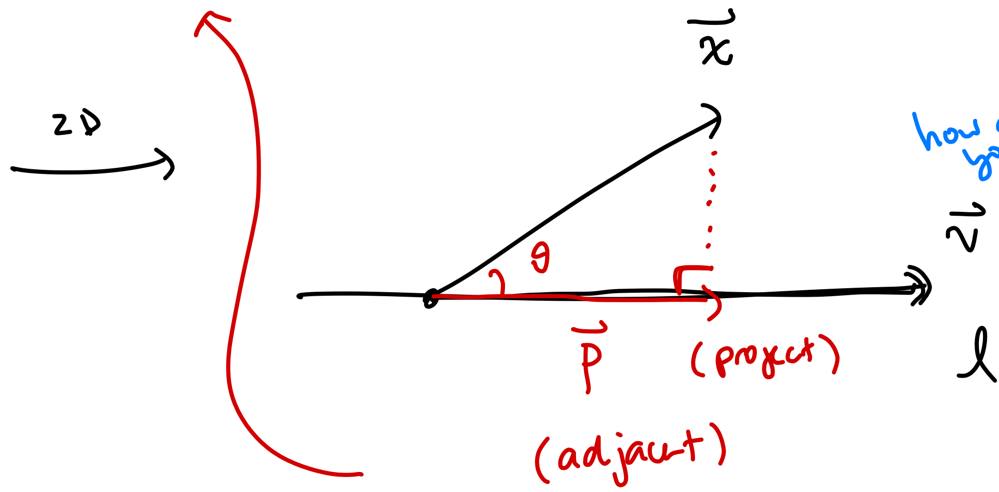
$$\boxed{p = \frac{x \cdot v}{||v||^2} v.}$$

how do you take dot product?

how do you find magnitude?






how do you scale $\vec v$ down to $\vec p$?

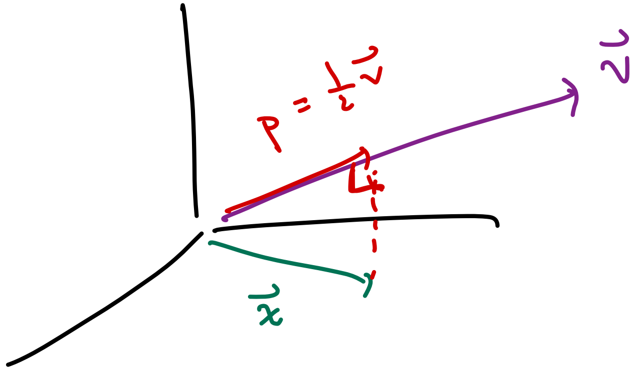
$|p| = |x| \cos\theta$

$$\vec p = \frac{x \cdot v}{||v||^2}\vec v = \underbrace{\frac{(2,3,1)\cdot(-1,1,0)}{||(-1,1,0)||^2}}_{\text{scalar}}\underbrace{(-1,1,0)}_{\text{vector}}$$

$$\vec{p} = \frac{x \cdot v}{\|v\|^2}\vec{v} = \underbrace{\boxed{\frac{(2,3,1)\cdot(-1,1,0)}{\|(-1,1,0)\|^{2}}}}_{\text{scalar}} \underbrace{\boxed{(-1,1,0)}}_{\text{vector}}$$

$$= \frac{2\cdot(-1) + 3\cdot 1 + 1\cdot 0}{\left(\sqrt{(-1)^2 + 1^2 + 0^2}\right)^{2}} (-1, 1, 0)$$

$$= \frac{-2+3+0}{1^2+1^2}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \frac{1}{2}\vec{v}$$



3. Find all matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad 2\times 2$$

which commute with

$$C = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \quad 2\times 2$$

Commute :

$AB = BA$

In general

$AB \neq BA$

What do a, b, c, d need to be in order for $\underbrace{MC = CM}_{\text{expand in terms of } a, b, c, d.}$?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} -a & a-b \\ -c & c-d \end{bmatrix} = \begin{bmatrix} -a+c & -b+d \\ -c & -d \end{bmatrix}$$

$-a = -a + c$

$a - b = -b + d$

$-c = -c$

$c - d = -d$

$$-a = -a + 2 \longrightarrow c = 0$$

$$a - b = -b + d \longrightarrow a = d$$

$$-c = -c \longrightarrow \text{nothing}$$

$$c - d = -d \longrightarrow c = 0$$

$a = d$
$c = 0$
no info about $\vec{b}$

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

commutes w/ $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$$MC = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -a & a-b \\ 0 & -a \end{bmatrix}$$

$$CM = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} -a & -b+a \\ 0 & -a \end{bmatrix}$$

✓

4. Find all matrices of the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ (called diagonal matrices), which commute with $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. If there are none, explain why.

v, w vectors

5. Let $v = \underline{(1, -1, 2, 3)}$ and $w = \underline{(0, 0, 2, 2)}$ be vectors in $\mathbb{R}^4$. Evaluate $\underbrace{|\vec{v} \cdot \vec{w}|}$ and $\underbrace{\|\vec{v}\|\,\|\vec{w}\|}$.

scalar quantities

4. Similar to #3

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

get some relationships between $x, y$.

5. More knowledge of dot product + magnitude

$$|v \cdot w| = |(1,-1,2,3) \cdot (0,0,2,2)| = |1 \cdot 0 + (-1) \cdot 0 + 2 \cdot 2 + 3 \cdot 2|$$

$$= |0 + 0 + 4 + 6| = 10$$

4. Find all matrices of the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ (called diagonal matrices), which commute with $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. If there are none, explain why.

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} x & 0 \\ y & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ x & y \end{pmatrix} \longrightarrow \begin{array}{l} x = x \\ 0 = 0 \\ y = x \\ y = y \end{array}$$

So to commute $x = y$ is necessary.

Indeed.

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ x & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

(Actually $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ commutes w/ everything.)

5. Let $v = (1, -1, 2, 3)$ and $w = (0, 0, 2, 2)$ be vectors in $\mathbb{R}^4$. Evaluate $|v \cdot w|$ and $\|v\|\,\|w\|$.

5. More knowledge of dot product + magnitude

$$|v \cdot w| = |(1,-1,2,3) \cdot (0,0,2,2)| = |1 \cdot 0 + (-1) \cdot 0 + 2 \cdot 2 + 3 \cdot 2|$$

$$= |0 + 0 + 4 + 6| = 10$$

$$\|v\| \cdot \|w\| = \sqrt{1^2 + (-1)^2 + 2^2 + 3^2}\,\sqrt{0^2 + 0^2 + 2^2 + 2^2}$$

$$= \sqrt{15}\,\sqrt{8} = \sqrt{120} = \sqrt{4 \cdot 30}$$

$$= 2\sqrt{30}$$

$$\boxed{\|v\|\,\|w\| \geq |v\ w|}$$

6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = \underset{3\times 3}{\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -3 \\ -1 & 1 & 1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}. = \underset{3\times 1}{\begin{pmatrix} -x + z \\ 2x - 3z \\ -x + y + z \end{pmatrix}}$$

Determine whether this transformation compresses or expands space, and whether it preserves orientation or reverses orientation.

det of $\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -3 \\ -1 & 1 & 1 \end{pmatrix}$ gives you all this information!

$|\det A| < 1 \longrightarrow$ compresses

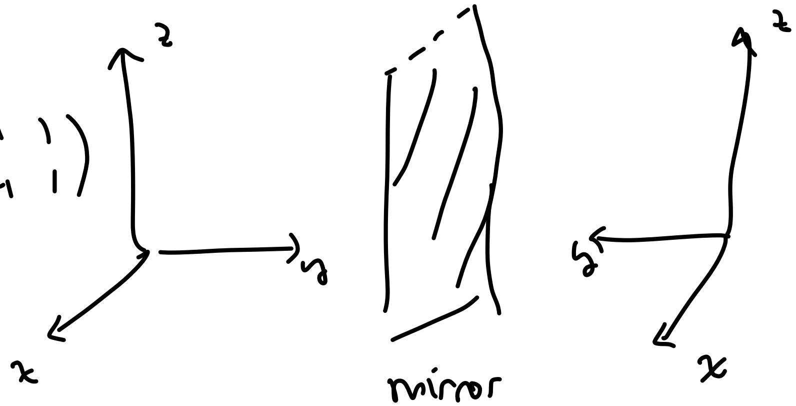
$|\det A| = 1 \longrightarrow$ preserves volume!

$|\det A| > 1 \longrightarrow$ expands

$\det A < 0$ reverses orientation

$\det A > 0$ preserves orientation

$$\det \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -3 \\ -1 & 1 & 1 \end{pmatrix} = -0 \det \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} + 0 \det \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} - 1 \det \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$$




$$= -1(1) = -1$$

$\det A < 0 \implies$ reverses

$|\det A| = 1$ preserves volume!

Derivatives in multi will be linear transformations. We can already see that a bit using the parametrization of a plane.

We'll do this later, not on midterm.