General Stuff

- Office Hours

  T: 12:30 - 1:30, Th: 10 - 11
- Quiz Thursday 2/11

  1 problem

  15 minutes to take exam

  5 minutes to upload to gradescope

  11:15 - 11:45 questions before quiz

  11:45 - 12:00 quiz

  12:00 - 12:05 uploading
- Lab after quiz from 12:20 - 1:10

Lab 2 due tonight! Exercises 1, 3

Continuity + derivatives

2.1 - 2.4

Level sets

Continuity, derivatives

parametrizations

1. Consider the function $f(x,y) = (x^2+y^2, \cos(xy), e^{x+y})$. (a) Find the domain and codomain of f. What size matrix is the derivative? (b) Find the total derivative $Df(x,y)$.

(a) $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

inputs domain

possible outputs codomain

(set of actual outputs range)

$f(x_1, \ldots, x_n) = (f_1, \ldots, f_m)$

n input m output

$f(x,y) = (x^2+y^2, \cos(xy), e^{x+y})$

2 inputs 3 outputs

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

2-vector 3-vector

$f(0,0) = (0,1,1)$

- $\mathbb{R}^2$ domain $\mathbb{R}^3$ codomain

- Df is a matrix

$f : \mathbb{R}^{\boxed{2}} \longrightarrow \mathbb{R}^{\boxed{3}}$

Swap!

Df is 3×2

$$Df = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \\ \partial f_3/\partial x & \partial f_3/\partial y \end{bmatrix}$$

(b)

$$f(x,y) = \Big(\underbrace{x^2+y^2}_{f_1},\ \underbrace{\cos(xy)}_{f_2},\ \underbrace{e^{x+y}}_{f_3}\Big)$$

$$Df = \begin{array}{c|cc} & & \\ f_1 & \boxed{\frac{\partial f_1}{\partial x}}^{2x} & \boxed{\frac{\partial f_1}{\partial y}}^{2y} \\ f_2 & \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ f_3 & \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \\ & x & y \end{array}$$

3×2

y is a constant x

$$\frac{\partial}{\partial x}\left(x^2+y^2\right) = \frac{\partial}{\partial x}\left(x^2\right) + \cancel{\frac{\partial}{\partial x}\left(y^2\right)}^{\,0} = 2x$$

$$\cos \cancel{5}^{\,y} x \rightarrow \cancel{5}_{\,y}\left(-\sin(\cancel{5}_{\,y} x)\right)$$

$$\frac{\partial}{\partial y}\left(x^2+y^2\right) = \cancel{\frac{\partial}{\partial y}\left(x^2\right)}^{\,0} + \frac{\partial}{\partial y}\left(y^2\right) = 2y$$

$$\frac{\partial}{\partial x}\left(\cos(xy)\right) = -y\sin(xy)$$

$$\frac{\partial}{\partial y}\left(\cos(xy)\right) = -x\sin(xy)$$

$$\frac{\partial}{\partial y}\left(e^{x+y}\right) = e^x \frac{\partial}{\partial y}\left(e^y\right) = e^{x+y}$$

$$\frac{\partial}{\partial x}\left(e^{x+y}\right) = \frac{\partial}{\partial x}\left(e^x e^y\right) = e^y \frac{\partial}{\partial x}\left(e^x\right) = e^y e^x = e^{x+y}$$

again.

$$Df(x,y) = \begin{bmatrix} 2x & 2y \\ -y\sin(xy) & -x\sin(xy) \\ e^{x+y} & e^{x+y} \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

multidimensional "slope" at $(0,0)$.

$Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$ is closest linear approximation to f at $(0,0)$.

$f(x,y) = (x^2+y^2, \cos(xy), e^{x+y})$

which linear transformation $T(x,y) = A\begin{pmatrix} x \\ y \end{pmatrix}$ best approximates f at $(0,0)$?

A is 3×2

A: $Df(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$.

at (x,y) $\begin{bmatrix} 2x & 2y \\ -y\sin & -x\sin \\ e^{x+y} & e^{x+y} \end{bmatrix}$

$f: \mathbb{R}^1 \to \mathbb{R}^1$ appr $J(x) = [m]x \cdot \left[\frac{df}{dx}\right]x$ the best matrix appox. to f.

$$\begin{bmatrix} \boxed{2x} & 2y \\ -y\sin(xy) & \boxed{-x\sin(xy)} \\ e^{x+y} & e^{x+y} \end{bmatrix}$$

$f(x,y) = (x^2+y^2,\ \underline{\cos(xy)},\ e^{x+y})$

$2x$ represents the slope in the x-direction of the first output x^2+y^2.

y-direction $2y$

2. Find the equation of the tangent plane to the equation $z = x^2 + y^2 + 3x$ at $(x, y) = (1, 2)$.

$$z = \underbrace{f(x_0, y_0)}_{\substack{\shortparallel \\ z_0}} + \boxed{\frac{\partial f}{\partial x}(x_0, y_0)}(x - x_0) + \boxed{\frac{\partial f}{\partial y}(x_0, y_0)}(y - y_0)$$

$8 \quad 5 \quad 1 \quad 4 \quad 2$

fancier point slope form

$$z - z_0 = m_x(x - x_0) + m_y(y - y_0) \qquad \text{pt slope}$$

$$z = f(x_0, y_0) + \underbrace{\left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right]}_{M} \underbrace{\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}}_{(x,y) - (x_0, y_0)}$$

$$z = f(x, y) = x^2 + y^2 + 3x \qquad (x_0, y_0) = (1, 2)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + 3x) = 2x + 0 + 3 = 2x + 3$$

$$Df = [2x + 3 \quad 2y]$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + 3x) = 0 + 2y + 0 = 2y$$

$$\nabla f = \begin{bmatrix} 2x + 3 \\ 2y \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(1, 2) = 2(1) + 3 = 5 \qquad \frac{\partial f}{\partial y}(1, 2) = 2 \cdot 2 = 4$$

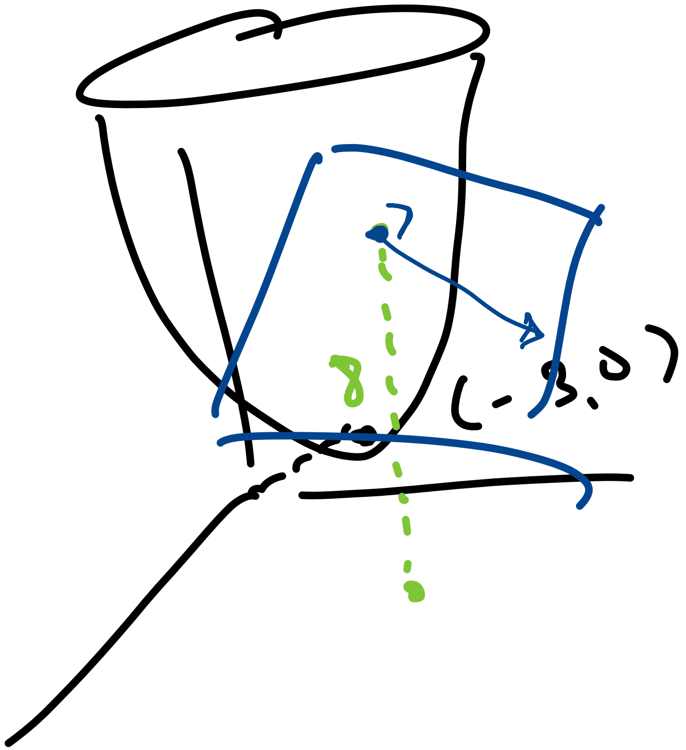
$$f(1, 2) = 1^2 + 2^2 + 3 \cdot 1 = 1 + 4 + 3 = 8$$

$$z = \underset{\substack{= \\ z_0}}{f(x_0,y_0)}^{\,8} + \underset{5}{\boxed{\frac{\partial f}{\partial x}(x_0,y_0)}}(x - \underset{1}{x_0}) + \underset{4}{\boxed{\frac{\partial f}{\partial y}(x_0,y_0)}}(y - \underset{2}{y_0})$$

$$z = 8 + 5(x-1) + 4(y-2)$$

$$z = \cancel{8} + 5x - 5 + 4y - \cancel{8} \longrightarrow \underline{5x + 4y - z = 5}$$

$n = (5, 4, -1)$



$f(x,y) = x^2 + y^2 + 3x$

$(5,4,-1)$ is perfectly $\perp$ to $z = x^2 + y^2 + 3x$ at $(1,2)$.

3. Determine whether the function $f(x, y) = \frac{x}{y} + \frac{y}{x}$ has continuous partials or not.

we have see whether

1D $\lim_{\boxed{x \to a}} f(x) = f(a)$

left right

2D $\lim_{\boxed{(x,y) \to (a,b)}} f(x,y) = f(a,b)$

all possible paths to (a,b)

$$\frac{\partial}{\partial x}\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{y}\frac{\partial}{\partial x}\cancel{(x)}^{1} + y\frac{\partial}{\partial x}\left(\frac{1}{x}\right)$$

$$= \frac{1}{y} + y\frac{-1}{x^2} = \frac{1}{y} - \frac{y}{x^2}$$

Similarly $\frac{\partial f}{\partial y} = \frac{y^2 - x^2}{xy^2}$

$\frac{\partial f}{\partial x} = \frac{x^2 - y^2}{x^2 y}$

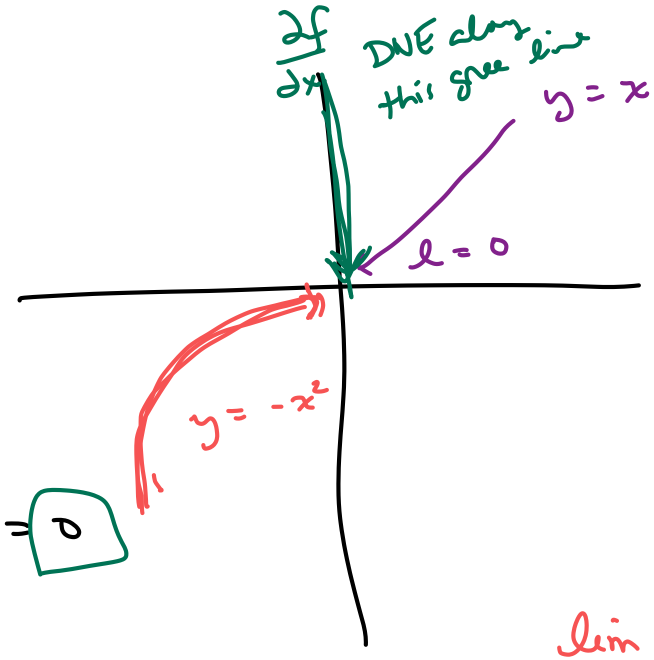
Are these functions cts? If I were to graph them, would have to "lift my pencil"?

$$\frac{\partial f}{\partial x} = \frac{x^2 - y^2}{x^2 y} \qquad \frac{\partial f}{\partial y} = \frac{y^2 - x^2}{y^2 x}$$

If $x \neq 0$, $y \neq 0$ then $f(x,y) = \frac{x}{y} + \frac{y}{x}$ and PA's are cts.

But if $(x,y) = (0,0)$, then we are dividing by 0 and we have issues.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 y}$$



$$\left\{ \begin{array}{l} (x,x) \longrightarrow (0,0) \\ x \to 0 \\ (0,y) \longrightarrow (0,0) \\ y \to 0 \end{array} \right.$$

$(x, -x^2) \longrightarrow (0,0)$

$x \to 0$

- $\underline{y = x}$

$$\lim_{(x,x)\to(0,0)} \frac{x^2 - x^2}{x^2 x} = \frac{0}{x^3} = \boxed{0}$$

- $\underline{\begin{array}{l} x = 0 \\ y \to 0 \end{array}}$

$$\lim_{(0,y)\to(0,0)} \frac{0^2 - y^2}{0 \cdot y} = \lim \frac{-y^2}{0} \quad \text{"} \quad \text{DNE}$$

lim won't exist if any two approaches give you different values!

lim $= 0$ and lim DNE do not agree so 2D limit doesn't exist!

not cts.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 y} \text{ DNE so } \frac{\partial f}{\partial x} \text{ is not cts!}$$

- $y = -x^2$

$(x, -x^2) \to (0,0)$

$$\lim_{(x,-x^2)\to(0,0)} \frac{x^2 - (-x^2)^2}{x^2(-x^2)} = \lim_{x\to 0} \frac{x^2 - x^4}{-x^4}$$

$$= \lim_{x\to 0} \frac{-1}{x^2} + 1 \qquad \boxed{\text{DNE}}$$

$$\text{"="} \quad \frac{-1}{0} + 1$$

:(

Getting a DNE means not cts.

4. Find the total derivative of the function $p(t) = (t, t^2, t^3)$. Does this function have a tangent plane at $(1, 1, 1)$?

5. Find the partial derivatives of $f(x, y) = \frac{x^2 y}{x^4+y^2}$. Are they continuous at the origin?

4. Find the total derivative of the function $p(t) = (t, t^2, t^3)$. Does this function have a tangent plane at $(1, 1, 1)$?

$p: \mathbb{R}^1 \to \mathbb{R}^3$ so D_p is 3×1, or a column vector.

$$D_p = \begin{bmatrix} \frac{d}{dt}(t) \\ \frac{d}{dt}(t^2) \\ \frac{d}{dt}(t^3) \end{bmatrix} = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}.$$

This function does <u>NOT</u> have a tangent plane!

Surfaces $z = f(x,y)$ or $f: \mathbb{R}^2 \to \mathbb{R}^1$ have tangent planes, but this is $p: \mathbb{R}^1 \to \mathbb{R}^3$.

It's a 1D curve.

5. Find the partial derivatives of $f(x, y) = \frac{x^2 y}{x^4+y^2}$. Are they continuous at the origin?

$$\frac{\partial f}{\partial x} = 2\left(\frac{-x^5 y + x y^3}{(x^4+y^2)^2}\right) \qquad \frac{\partial f}{\partial y} = \frac{x^6 - x^2 y^2}{(x^4+y^2)^2}$$ by quotient rule!

These are not cts! If we approach the origin (divide by 0 issues) at different slopes, we get different answers.

$y = mx$ $\quad (x, mx) \to (0,0)$

$$\lim_{(x,mx)\to(0,0)} \frac{\partial f}{\partial x} = 2\left(\frac{-x^5(mx) + x(mx)^3}{(x^4+(mx)^2)^2}\right) = 2\lim_{x\to 0} \frac{-mx^6 + m^3x^4}{(x^4+m^2x^2)^2}$$

$$= 2\lim_{x\to 0} \frac{-mx^6 + m^3x^4}{x^4(x^2+m^2)^2} = 2\lim_{x\to 0} \frac{-mx^2+m^3}{(x^2+m^2)^2} = \frac{m^3}{m^4}$$

$$= \frac{1}{m}.$$ This depends on m

$$\neq \frac{\partial f}{\partial x}(0,0).$$ So the 2D limit doesn't exist!

$y = mx$

$$\lim_{(x,mx)\to(0,0)} \frac{\partial f}{\partial y} = \lim_{(x,mx)\to(0,0)} \frac{x^6 - x^2y^2}{(x^4+y^2)^2} = \lim_{x\to 0} \frac{x^6 - x^2(mx)^2}{(x^4+(mx)^2)^2}$$

$$= \lim_{x\to 0} \frac{x^6 - m^2x^4}{(x^4+m^2x^2)^2} = \lim_{x\to 0} \frac{x^6 - m^2x^4}{x^4(x^2+m^2)^2}$$

$$= \lim_{x\to 0} \frac{x^2 - m^2}{(x^2+m^2)^2} = \frac{-m^2}{m^4} = \frac{-1}{m^2}$$

This depends on m so the limit DNE also?

Neither $\frac{\partial f}{\partial x}$. $\frac{\partial f}{\partial y}$ IS cts.