General Stuff

- Office Hours

  T: 12:30 - 1:30, Th: 10 - 11

  So office hours after class today
- Lab 08 due by the end of tonight
- Quiz 5 on Thursday 3/25
- Topics include 8.1 and 8.3

  1 problem

  15 minutes to take quiz

  5 minutes to upload to gradescope

  11:15 - 11:45 questions before quiz

  11:45 - 12:00 quiz

  12:00 - 12:05 uploading

1. Let $c(t)$ be a path in $\mathbb{R}^n$ and $T(t)$ the unit tangent vector of $c(t)$ at time t. Compute

$$\int_c T \cdot ds.$$

vector line integral!

$c(t)$ is some curve, $a \le t \le b$.

$T(t) = \text{unit } \underbrace{\text{tangent vector of } c(t)}$

$= \underbrace{\text{unit}} \text{ of } \underbrace{c'(t)}$

$$T(t) = \frac{c'(t)}{\|c'(t)\|}$$

$$\int_c T \cdot d\vec{s} = \underbrace{\int_a^b T(t) \cdot c'(t)\, dt}_{+1} \checkmark = \int_a^b \underbrace{\frac{c'(t)}{\|c'(t)\|}}_{+1} \cdot c'(t)\, dt$$

$$= \int_a^b \frac{1}{\|c'(t)\|} \big(\underbrace{c'(t) \cdot c'(t)} \big)\, dt$$

Recall

$\underbrace{v \cdot v = \|v\|^2}$

$v \cdot v = v_1^2 + v_2^2 + \dots + v_n^2$

$= \|v\|^2$

$$= \int_a^b \frac{1}{\|c'(t)\|} \underbrace{\|c'(t)\|^2}\, dt$$

scalar line integral

$$= {}^{+1}\int_a^b 1\, \underbrace{\|c'(t)\|\, dt}_{ds} = \int_C 1\, ds$$

$= \text{arc length of } C$ +1

$\square$

Compute $\rightsquigarrow$ scalar line

2. a) ~~Copute~~ the path integral of $f(x, y) = y^2$ over the curve $y = \sqrt{1-x^2}$ from $-1 \leq x \leq 1$.
b) Consider the reparametrization $c(\theta) = (\cos(\theta), \sin(\theta))$ from $0 \leq \theta \leq \pi$. Recompute the integral.

a) Consider the graph parametrization $p(t) = (t, \sqrt{1-t^2}) \quad -1 \leq t \leq 1$.

$$\int_C y^2 \, ds = \int_{-1}^{1} f(p(t)) \, \|p'(t)\| \, dt \rightsquigarrow \frac{1}{\sqrt{1-t^2}}$$

$(1-t^2)^{1/2}$

$$p'(t) = \frac{d}{dt}\left(t, \sqrt{1-t^2}\right) = \left(1, \frac{1}{2}(1-t^2)^{-1/2} \cdot (-2t)\right)$$

$$= \left(1, \frac{-t}{\sqrt{1-t^2}}\right)$$

$$\|p'(t)\| = \sqrt{1^2 + \left(\frac{-t}{\sqrt{1-t^2}}\right)^2} = \sqrt{1 + \frac{t^2}{1-t^2}}$$

$$= \sqrt{\frac{1-t^2}{1-t^2} + \frac{t^2}{1-t^2}} = \sqrt{\frac{1-\cancel{t^2}+\cancel{t^2}}{1-t^2}}$$

$$= \sqrt{\frac{1}{1-t^2}} = \underbrace{\frac{1}{\sqrt{1-t^2}}}$$

Speed at which $p(t)$ is traveling at time t

y^2 $\quad y = \sqrt{1-t^2}$

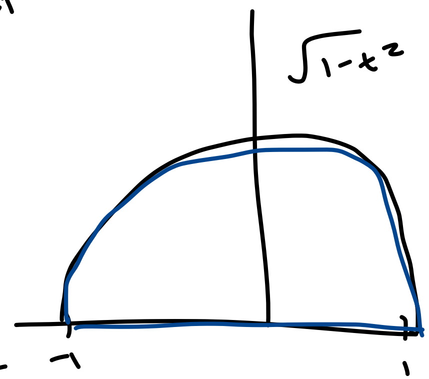
$$\int_C y^2\, ds = \int_{-1}^{1} f(p(t))\, \|p'(t)\|\, dt = \int_{-1}^{1} (1-t^2)\frac{1}{\sqrt{1-t^2}}\, dt$$

$$= \int_{-1}^{1} (1-t^2)(1-t^2)^{-1/2}\, dt = \int_{-1}^{1} (1-t^2)^{1/2}\, dt$$

$t = \sin\theta$

!!

$$= \int_{-1}^{1} \sqrt{1-t^2}\, dt = \text{Area of half of a circle of rad } 1 = \frac{\pi}{2}$$



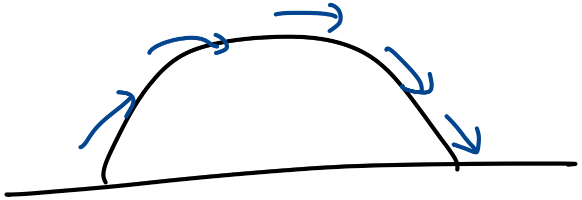
2. a) Copute the path integral of $f(x,y) = y^2$ over the curve $y = \sqrt{1-x^2}$ from $-1 \le x \le 1$.
b) Consider the reparametrization $c(\theta) = (\cos(\theta), \sin(\theta))$ from $0 \le \theta \le \pi$. Recompute the integral.

b) $c(\theta) = (\cos(\theta), \sin\theta) \qquad 0 \le \theta \le \pi$

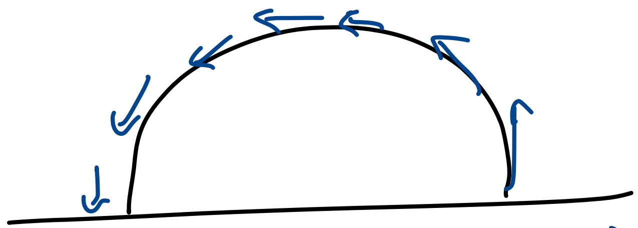
Since $c(\theta)$ is a reparametrization of $p(t) = (t, \sqrt{1-t^2})$

then $$\int_0^{\pi} f(c(\theta)) \underbrace{\|c'(\theta)\|}_{\text{speed, no direction}} d\theta = \int_c y^2 \, ds = \frac{\pi}{2}.$$

Note: Vector line integrals are affected by orientation



$p(t) = (t, \sqrt{1-t^2})$



$c(\theta) = (\cos\theta, \sin\theta)$

vector line integrals would had opposite sign, but scalar line integrals don't have this issue

$$\int_C y^2\, ds = \int_0^\pi \sin^2\theta \underbrace{\|c'(\theta)\|}_{1} d\theta \qquad y = \sin\theta$$

$$c(\theta) = (\cos\theta, \sin\theta) \qquad c'(\theta) = (-\sin\theta, \cos\theta)$$

$$\|c'(\theta)\| = \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1$$

$$= \int_0^\pi \sin^2\theta\, d\theta = \int_0^\pi \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \int_0^\pi \frac{1}{2}\, d\theta - \underbrace{\int_0^\pi \cancel{\frac{\cos(2\theta)}{2}}\, d\theta}_{0} = \frac{\pi}{2}$$

$$\cos^2\theta + \sin^2\theta = 1$$

Trick:

$$\int_0^\pi \sin^2\theta\, d\theta = \int_0^\pi \cos^2\theta\, d\theta = \frac{\pi}{2} \qquad \int_0^\pi 1\, d\theta = \pi$$

$$\int_0^\pi \cos^2\theta + \sin^2\theta \, d\theta = \pi,$$ but since $0 \to \pi$ is a period of $\cos^2$, $\sin^2$, they contribute equally.

$$\Longrightarrow \quad \int_0^\pi \cos^2\theta \, d\theta = \int_0^\pi \sin^2\theta \, d\theta = \frac{\pi}{2}.$$

3. Which of the following vector fields are conservative?

- $F(x, y, z) = (y - \sin(x)e^y z, \cos(x)e^y z + x, \cos(x)e^y)$
- * $G(x, y, z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$
- $H(x, y, z) = (2xy + z^3, x^2, 3xz^2)$

Integrate the vector field H over the curve $p(t) = (t, -t^2, t)$ from $t = -1$ to $t = 1$.

$\to$ $F = \nabla \phi$

easiest to calculate $\to$ * $\nabla \times F = 0$

$\oint_C F \cdot ds = 0$ C closed loop!

F defined except at finite pts.

- By the note, it's easiest to just check that $\nabla \times F = 0$.

$$\nabla \times \left(y - \sin(x)e^y z,\ \cos(x)e^y z + x,\ \cos(x)e^y \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - \sin(x)e^y z & \underbrace{\cos(x)e^y}\, z + x & \cos(x)e^y \end{vmatrix}$$

$$= \Big(\cancel{\cos(x)e^y - \cos(x)e^y}_{0},\ \cancel{-\sin(x)e^y - (-\sin(x))e^y}_{0},$$

$$\cancel{-\sin(x)e^y z + 1 - 1 - \sin(x)e^y z}_{0} \Big) = (0,0,0)$$

Since F is $\underbrace{\text{defined everywhere}}$ and $\nabla \times F = 0$, then F is conservative! $\Rightarrow \oint_C F \cdot ds = 0$

- Again, $G(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$.

$\nabla \times G = (0, 0, 0)$. But, this is not conservative!

G is not defined at $(0,0)$, so if you integrate in a loop around $(0,0)$, $\oint \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right) \cdot ds \neq 0$

Not conservative!

3. Which of the following vector fields are conservative?

- $F(x, y, z) = (y - \sin(x)e^y z, \cos(x)e^y z + x, \cos(x)e^y)$
- $G(x, y, z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$
- $H(x, y, z) = (2xy + z^3, x^2, 3xz^2)$

Integrate the vector field H over the curve $p(t) = (t, -t^2, t)$ from $t = -1$ to $t = 1$.

$H(x,y,z) = (2xy + z^3, x^2, 3xz^2)$

H is defined everywhere.

$\nabla \times H = (0 - 0, 3z^2 - 3z^2, 2x - 2x) = (0,0,0)$

It's conservative!

Trick: $\int_p H \cdot ds = \phi(b) - \phi(a)$ where $\nabla\phi = H$.

$$H(x,y,z) = (\underbrace{2xy + z^3}_{\frac{\partial}{\partial x}\phi},\ \underbrace{x^2}_{\frac{\partial}{\partial y}\phi},\ \underbrace{3xz^2}_{\frac{\partial}{\partial z}\phi})$$

??

$$\int 2xy + z^3\, dx = \boxed{\boxed{x^2y + xz^3}} + f(\cancel{y,z})$$

0

??

Solve it by looking at it!

$$\int x^2\, dy = x^2y + g(\cancel{x,z})$$

xz^3

??

$$\int 3xz^2\, dz = xz^3 + h(\cancel{x,y})$$

x^2y

In fact $\phi = x^2y + xz^3$ $\quad \nabla\phi = H = (2xy + z^3,\ x^2,\ 3xz^2)$.

plug in $p(t) = (t, -t^2, t)$

shortcut since H is conservative

$$\int_p H \cdot ds = \int_{-1}^{1} (2xy + z^3,\ x^2,\ 3xz^2) \cdot p'(t)\, dt = \phi(p(1)) - \phi(p(-1))$$

$$= \phi(1,-1,1) - \phi(-1,-1,-1)$$

$$= (1^2(-1) + 1) - ((-1)^2(-1) + (-1)(-1)^3) = 0$$

4. Let C be the closed curve $c(t) = (3 + 2\cos(t), -2 + 3\sin(t))$ from $0 \le t \le 2\pi$. Compute the line integral

$$\int_C y^2 z e^{xyz}\, dx + e^{xyz}(xyz + 1)\, dy + xy^2 e^{xyz}\, dz.$$