General Stuff

- Office Hours

  T: 12:30 - 1:30, Th: 10 - 11

- Midterm 5 Thursday 4/1
- Topics include 8.1, 6.1, 6.2

  2 problems

  30 minutes to take quiz

  ~~5~~ minutes to upload to gradescope

  11:15 - 11:25 questions before quiz

  11:25 - 11:55 quiz

  11:55 - 12:05 uploading

- Lab 09 due tonight!

change of variables
- polar
- linear

Green's thm

5 - 10

1. Let B be the unit box with boundary ∂B. Denote the clockwise direction of the boundary by ∂B_{cw}. Evaluate the line integral

$$\int_{\partial B_{\mathrm{cw}}} y^2\,dx - x^2\,dy.$$

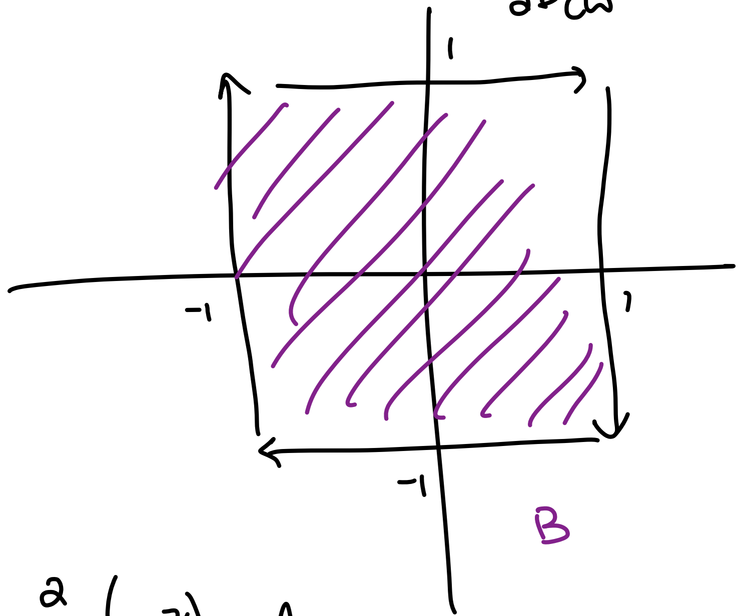
Green's Theorem

$$\oint_{\partial W} (P, Q) \cdot d\vec{s} = \iint_W \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

counter clockwise

$=$

$$- \oint_{\partial W_{cw}} (P, Q) \cdot d\vec{s}$$



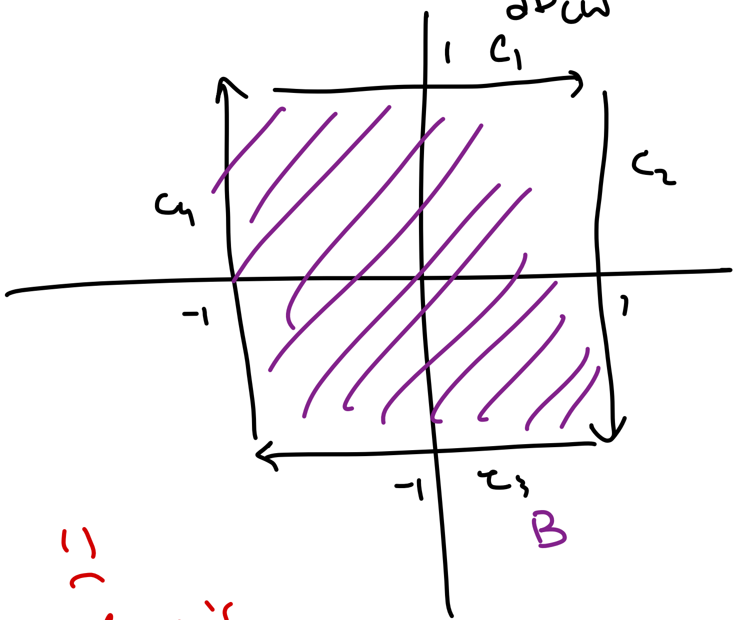
because ∂B_{cw} is clockwise

$$\oint_{\partial B_{cw}} y^2\, dx - x^2\, dy = - \iint_B \frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial y}(y^2)\, dA$$

$$= -\int_{-1}^{1}\int_{-1}^{1} -2x - 2y \, dx\, dy$$

$$= -\int_{-1}^{1} \left(-x^2 - xy\right)\Big|_{-1}^{1} dy = -\int_{-1}^{1} (-1-y) - (-1+y)\, dy$$

$$= -\int_{-1}^{1} -2y\, dy = 2\int_{-1}^{1} y\, dy = 0$$



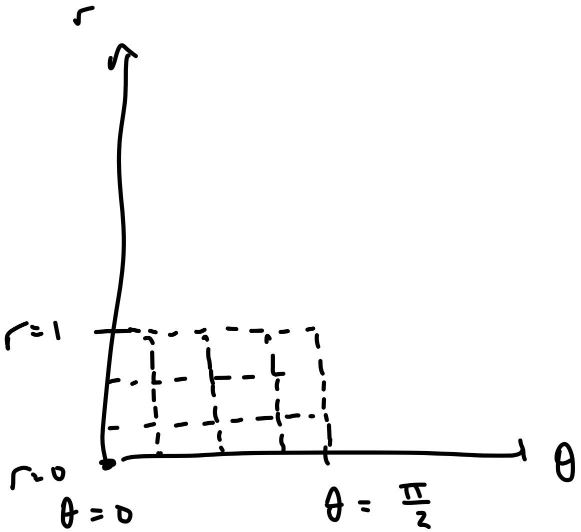
$$\int_{\partial B_{cw}} (y^2, -x^2)\cdot d\vec{s} = \int_a^b F(c(t))\cdot c'(t)\, dt$$

$$= \int_{c_1} F + \int_{c_2} F + \int_{c_3} F + \int_{c_4} F$$

$\overset{!!}{\frown}$ Green's theorem helps!

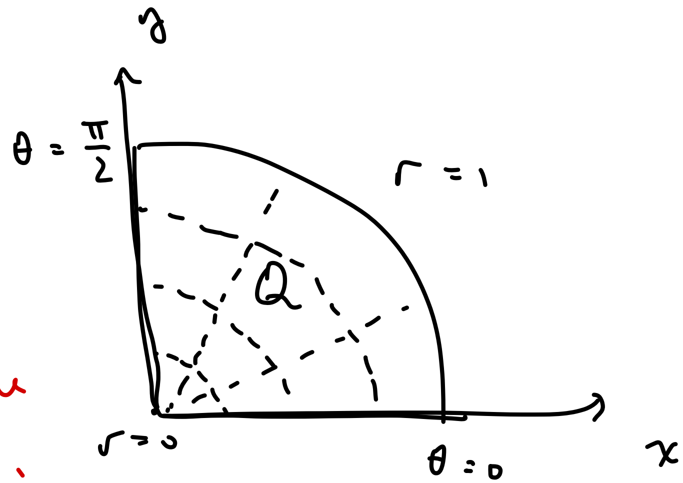
2. Let Q be the quarter unit circle from $0 \leq \theta \leq \pi/2$. Find the integral

$$\iint_Q e^{-(x^2+y^2)} dA.$$



$x = r\cos\theta$

$y = r\sin\theta$



not integrable in cartesian!

$$\iint_Q e^{-x^2-y^2} dA = \int_0^1 \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2} dA$$

memorize r for polar coordinates

"$dx\,dy = r\,dr\,d\theta$"

$dA = r\,dr\,d\theta$

$$= \int_0^{\pi/2} \int_0^1 e^{-(r\cos\theta)^2 - (r\sin\theta)^2} \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta$$

Jacobian

$$= \int_0^{\pi/2} \int_0^1 e^{-(r\cos\theta)^2 - (r\sin\theta)^2} \, r \, dr \, d\theta$$

no θ's in integrand

$$= \frac{\pi}{2} \int_0^1 e^{-r^2(\cos^2\theta + \sin^2\theta)} \, r \, dr$$

$$= \frac{\pi}{2} \int_0^1 e^{-r^2} r \, dr = \frac{\pi}{2} \int e^u \left(\frac{-1}{2} du \right)$$

$u = -r^2$

$du = -2r \, dr$

$\frac{-1}{2} du = r \, dr$

$$= \frac{-\pi}{4} \int e^u \, du = \frac{-\pi}{4} \left(e^{-r^2} \right)\Big|_0^1$$

$$= \frac{-\pi}{4} \left(e^{-1} - 1 \right)$$

$$= \frac{\pi}{4} \left(1 - \frac{1}{e} \right)$$

no negative sign necessary!

3. Let D be the unit circle with counter clockwise boundary ∂D. Compute the integral

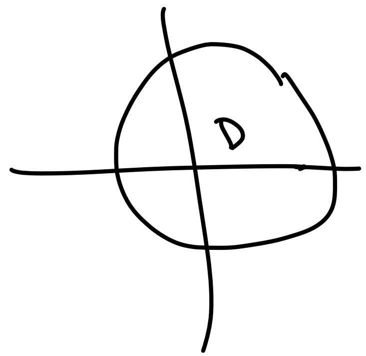
$$\int_{\partial D} -y^3\,dx + x^3\,dy.$$

$F = (-y^3, x^3)$

$c(t) = (\cos\theta, \sin\theta)$

$$\int_{\partial D} -y^3\,dx + x^3\,dy = \int_0^{2\pi} -\sin^3\theta\,(\sin\theta)\,d\theta + \cos^3\theta\,(-\sin\theta)\,d\theta$$

$$= \int_0^{2\pi} -\sin^4 + \cos^4\,d\theta$$:($= \frac{3\pi}{2}$



Green's theorem!

$$\underset{\text{CCW}}{=} (+)\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 3x^2 + 3y^2\,dA$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3x^2+3y^2\,\underbrace{dy\,dx}$$

$$= \int_0^{2\pi}\int_0^1 3(r\cos\theta)^2 + 3(r\sin\theta)^2 \, \underbrace{r \, dr \, d\theta}$$

$$= \int_0^{2\pi}\int_0^1 3r^2 \left(\cancelto{1}{\cos^2\theta + \sin^2\theta} \right) r \, dr \, d\theta$$

$$= 2\pi \int_0^1 3r^3 \, dr = 2\pi \left(\frac{3}{4} r^4 \right)\Big|_0^1 =$$

$$= \frac{6\pi}{4} = \frac{3\pi}{2}$$

$1 \leq u \leq 3$ $\quad$ $5 \leq v \leq 10$

4. Let W be the region bounded by $\left\{1 \leq 2x - y \leq 3\right\}$ and $5 \leq x + 2y \leq 10$. Compute the integral

$$\iint_W y\,dA.$$

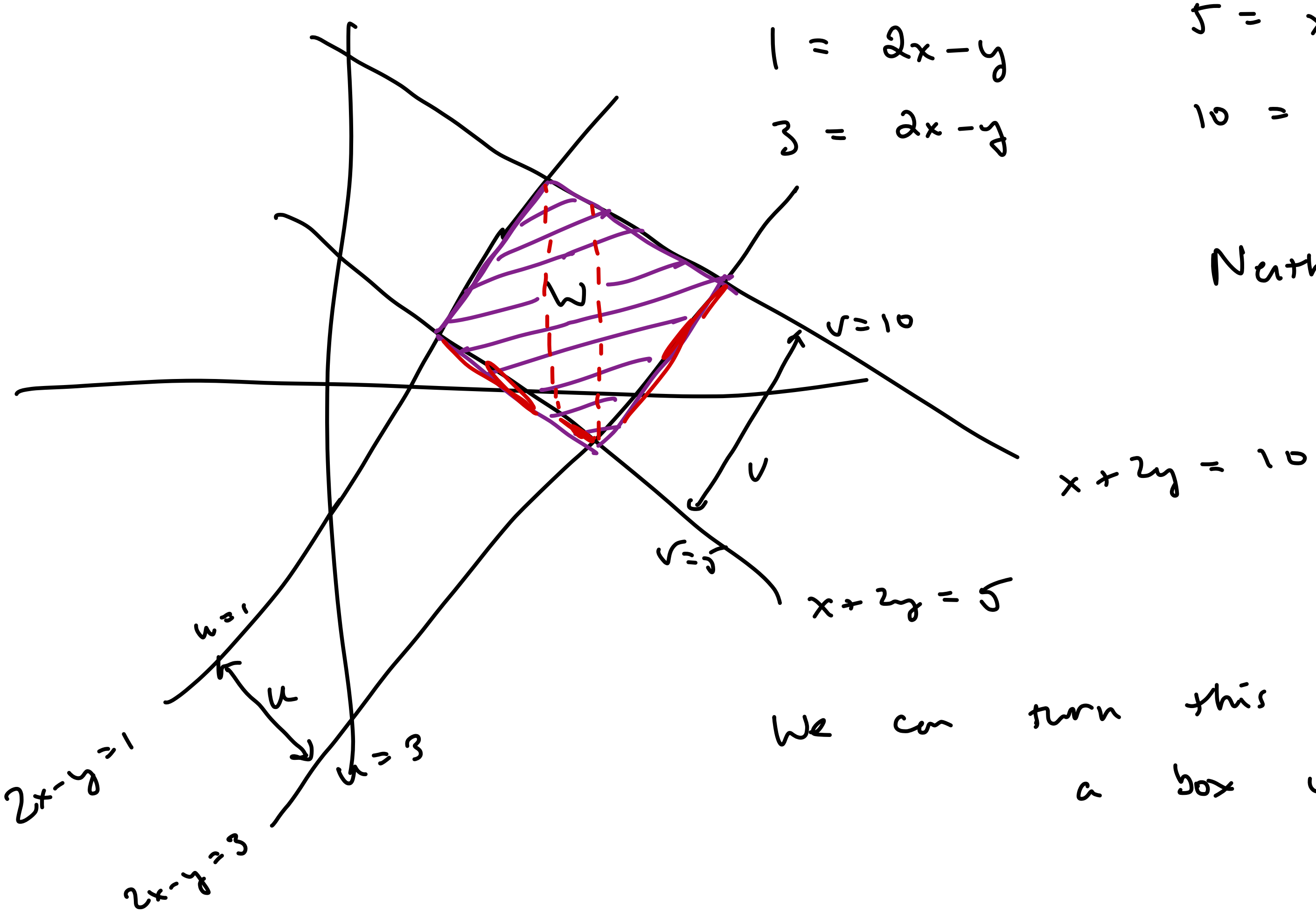
Change of variable problem!

$1 = 2x - y \qquad y = 2x - 1$

$3 = 2x - y \qquad y = 2x - 3$

$5 = x + 2y \qquad y = -\frac{1}{2}x + \frac{5}{2}$

$10 = x + 2y \qquad y = -\frac{1}{2}x + \frac{10}{2}$



Neither x-simple nor y-simple

We can turn this parallelogram into a box using change of variables!

$u = 2x - y$

$v = x + 2y$

But!

$$\iint_W y \, dA = \iint y \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

u,v?

Solve for y?

x and y in terms of u,v?

So we need to solve for x,y!

$u = 2x - y$

$v = x + 2y$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{pmatrix} x = r\cos\theta \\ y = r\cos\theta \end{pmatrix}$$

$$\underbrace{\begin{aligned} x &= \tfrac{2}{5}u + \tfrac{1}{5}v \\ y &= \tfrac{-1}{5}u + \tfrac{2}{5}v \end{aligned}}_{\text{Old vars in terms of new variables}}$$

$$\left.\begin{aligned} u &= 2x - y \\ v &= x + 2y \end{aligned}\right\}$$

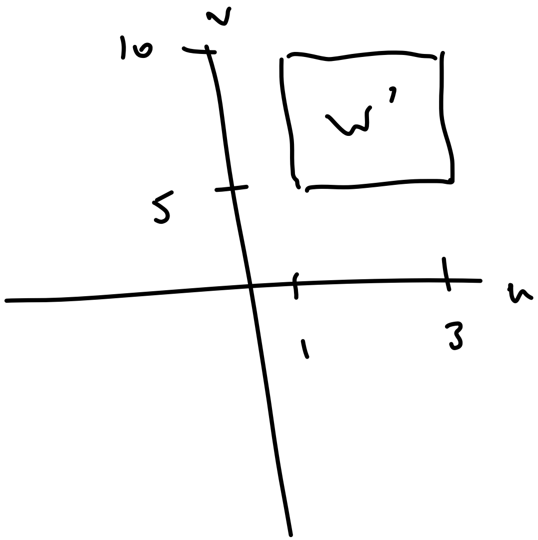
$$1 \le \underbrace{2x-y}_{u} \le 3$$

$$5 \le \underbrace{x+2y}_{v} \le 10$$

$$1 \le u \le 3$$

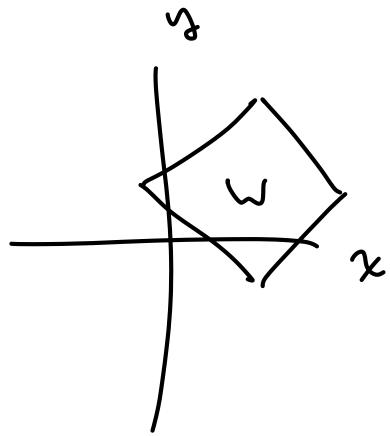
$$5 \le v \le 10$$

$$\iint_W y\, dA = \iint_{W'} \left(\tfrac{-1}{5}u + \tfrac{2}{5}v\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$



$$x = \tfrac{2}{5}u + \tfrac{1}{5}v$$

$$y = \tfrac{-1}{5}u + \tfrac{2}{5}v$$



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"They cancel"

$$= \int_5^{10}\int_1^3 -\frac{1}{5}u + \frac{2}{5}v \left| \frac{\partial(x,y)^*}{\partial(\cancel{u,v})} \right|_{\frac{1}{5}} \cancel{du}\, dv$$

col.

$$F(u,v) = \left(\overset{x}{\frac{2}{5}u + \frac{1}{5}v}, \overset{y}{-\frac{1}{5}u + \frac{2}{5}v}\right)$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det D$$

$$\frac{\partial x^*}{\partial u^*} = \frac{2}{5} \qquad \frac{\partial x}{\partial v} = \frac{1}{5}$$

$$\frac{\partial y}{\partial u} = \frac{-1}{5} \qquad \frac{\partial y}{\partial v} = \frac{2}{5}$$

$$\text{Jac} = |\det(DF)|$$

$$= \left|\det\begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix}\right|$$

$$= \left|\frac{4}{25} + \frac{1}{25}\right| = \left|\frac{5}{25}\right| = \frac{1}{5}$$

$$u = 3x$$

$$du = 3\,dx$$

$\downarrow$

1D Jacobian

$$= \int_5^{10}\int_1^3 -\frac{1}{5}u + \frac{2}{5}v \left(\frac{1}{5}\right) du\, dv$$

$$= \frac{1}{25}\int_5^{10}\int_1^3 -u + 2v \, du \, dv$$

$$= \frac{1}{25}\int_5^{10} \left(-u^2 + 2uv\right)_1^3 \, dv$$

$$= \frac{1}{25}\int_5^{10} (-9 + 6v) - (-1 + 2v) \, dv$$

$$= \frac{1}{25}\int_5^{10} -8 + 4v \, dv = \frac{1}{25}\left(-8v + 2v^2\right)_5^{10}$$

$$= \frac{1}{25}\left((-80 + 200) - (-40 + 50)\right)$$

$$= \frac{1}{25}\left(120 - 10\right) = \frac{110}{25} = \frac{22}{5}$$