General Stuff

- Office Hours

  T: 12:30 - 1:30, Th: 10 - 11
- Midterm 6 Thursday 4/13
- Topics include 7.3 - 7.5 (?)

  2 problems

  30 minutes to take quiz

  5 minutes to upload to gradescope

  11:15 - 11:25 questions before quiz

  11:25 - 11:55 quiz

  11:55 - 12:05 uploading
- Lab 10 due tonight!

1. Consider the parametrized surface given by

$$\Phi(u, v) = (2v\cos(u), 2v\sin(u), v^2).$$

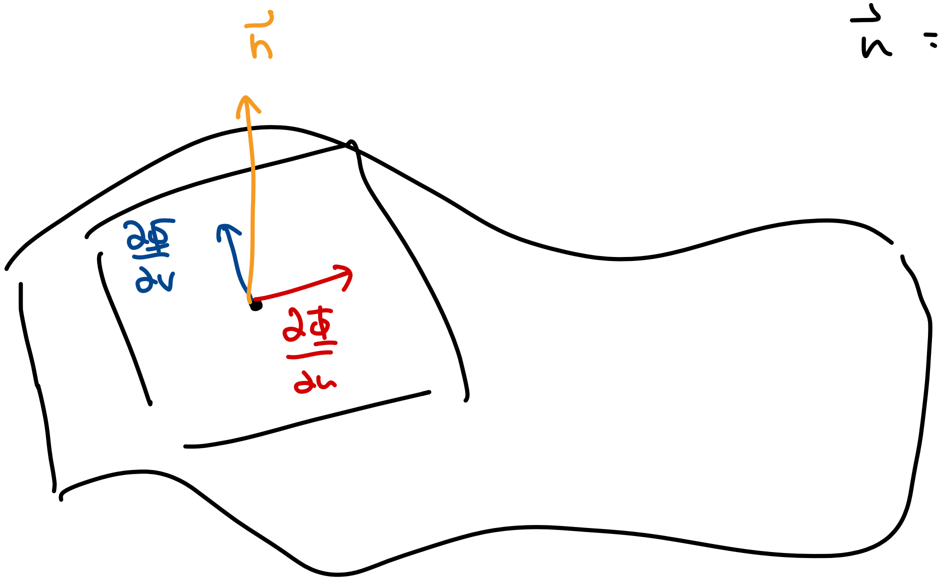
Find the tangent plane at the point corresponding to $(u_0, v_0) = (\pi, 2)$. (Bonus: what is the cartesian equation $z = f(x, y)$ for this surface?)

Tangent plane eq'n for a parametrized surface

$$\vec{n}\,(\vec{x} - \vec{a}) = 0 \qquad a = \Phi(u_0, v_0).$$

$$\left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right)(u_0, v_0)\left(\vec{x} - \Phi(u_0, v_0)\right) = 0 \qquad \vec{x} = (x, y, z)$$

$$\vec{n} = \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \quad *$$



each direction vectors.

$$\Phi(u,v) = (2v\cos(u),\ 2v\sin(u),\ v^2).$$

$$a = \Phi(u_0,v_0) = \Phi(\pi,2)$$

$$= (2\cdot 2\cos(\pi),\ 2\cdot 2\sin(\pi),\ 2^2)$$

$$= \underline{(-4,0,4)} =$$

$$\frac{\partial \Phi}{\partial u} = (-2v\sin(u),\ 2v\cos(u),\ 0)$$

$$\frac{\partial \Phi}{\partial v} = (2\cos(u),\ 2\sin(u),\ 2v)$$

direction vectors for tangent plane at (u,v)

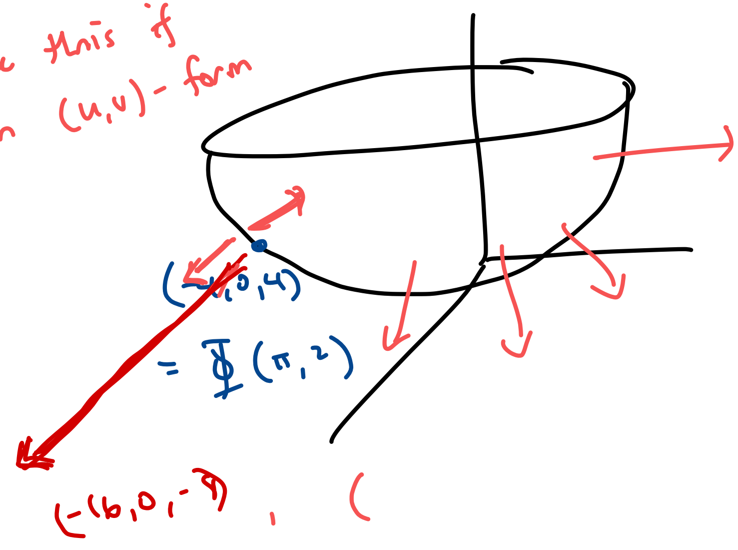
$\vec{n}(u,v)$ = normal vector to the surface Φ at (u,v)

$$= \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2v\sin(u) & 2v\cos(u) & 0 \\ 2\cos(u) & 2\sin(u) & 2v \end{vmatrix}$$

$$= \left(4v^2\cos(u),\ 4v^2\sin(u),\ -4v\sin^2(u) - 4v\cos^2(u)\right)$$

$$= \underline{4\left(v^2\cos(u),\ v^2\sin(u),\ -v\right)}$$

* keep n like this if in (u,v)-form



The base point is $(u_0, v_0) = (\pi, 2)$

$$n(\pi,2) = 4\left(2^2\cos(\pi),\ 2^2\sin(\pi),\ -2\right)$$

$$= 4(-4, 0, -2) = (-16, 0, -8) \longrightarrow \underline{(-2,0,-1)} \text{ or } \underline{(2,0,1)}$$

only w/ numbers!

$$\vec{n}\cdot(\vec{x} - \vec{a}) = 0 \qquad (-16,0,-8)\cdot\left((x,y,z) - (-4,0,4)\right) = 0$$

$$-16(x+4) + -8(z-4) = 0$$

$$-2(x+4) - (z-4) = 0$$

$$-2x - z = 4$$

$$\underline{2x + z = -4}$$

$$\frac{\partial \Phi}{\partial u} = \left(-2v\sin(u),\ 2v\cos(u),\ 0 \right)$$

$$\frac{\partial \Phi}{\partial u}(\pi, 2) = (0, -4, 0)$$

$$\frac{\partial \Phi}{\partial v} = \left(2\cos(u),\ 2\sin(u),\ 2v \right)$$

$$\frac{\partial \Phi}{\partial v}(\pi, 2) = (-2, 0, 4)$$

$$n = \begin{vmatrix} i & j & k \\ 0 & -4 & 0 \\ -2 & 0 & 4 \end{vmatrix} = \left(-16, 0, -8 \right) \quad @ \ (\pi, 2)$$

specifically

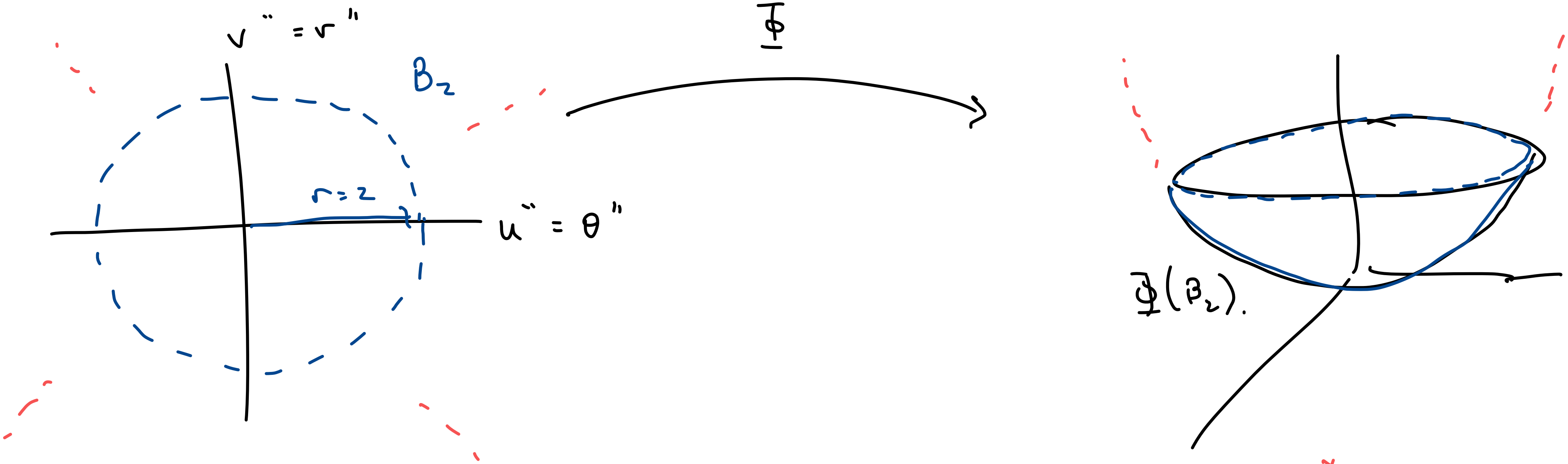
Bonus : $z = \frac{1}{2}(x^2 + y^2)$

the thing inside cos is angle

2. Let S be the same surface as before

$$\Phi(u, v) = (2\overset{r}{\underline{v}}\cos(\overset{\theta}{\underline{u}}), 2v\sin(u), v^2)$$

and let B_2 be the ball of radius $r = 2$ in the uv-plane. Find the area of the surface $\Phi(B_2)$.



$$\text{Area of surface} = \iint_{\Phi(B_2)} 1 \, dS = \iint_{B_2} 1 \, \overset{*}{\underbrace{\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|}} \, du \, dv$$

"= det J"

= area of dudv

$$\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} = 4\left(v^2 \cos(u),\ v^2 \sin(u),\ -v \right)$$

$$\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| = \sqrt{16\left(v^4 \cos^2 u + v^4 \sin^2 u + v^2 \right)}$$

$$= 4\sqrt{v^4\left(\cos^2(u) + \sin^2(u)\right) + v^2}$$

$$= 4\sqrt{v^4 + v^2} = 4v\sqrt{v^2+1} = \text{magnitude of } n \text{ at } (u,v)$$

$$\text{Area} = \iint_{\Phi(B_2)} 1\, dS = \iint_{B_2} \underbrace{\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|} \, du\, dv$$

$$= \iint_{B_2} 4v\sqrt{v^2+1}\ du\, dv \qquad B_2 = \left\{ (u,v) \;\middle|\; \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le 2 \end{array} \right.$$

$$u = \theta \qquad B_2 = \begin{cases} r & 0 \to 2 \\ \theta & 0 \to 2\pi \end{cases}$$

$$v = r \qquad = \begin{cases} u & 0 \to 2\pi \\ v & 0 \to 2 \end{cases}$$

$$= \int_0^{2\pi}\int_0^2 4v\sqrt{v^2+1}\; dv\, du = 2\pi \int_0^2 4v\sqrt{v^2+1}\; dv$$

$$\alpha = v^2+1$$

$$d\alpha = 2v\, dv \Rightarrow \quad 4v\, dv = 2\, d\alpha$$

$$= 2\pi \int \sqrt{\alpha}\, 2\, d\alpha = 2\pi \left(2\alpha^{3/2}\right)$$

$$= 2\pi \left(2\left(v^2+1\right)^{3/2} \right)_0^2 = 2\pi\left(2\cdot 5^{3/2} - 2\cdot 1^{3/2}\right)$$

$$= 2\pi\left(2\sqrt{125} - 2\right) = 4\pi\left(\sqrt{125} - 1\right)$$

3. Compute the scalar surface integral

$$\iint_{\Phi(B_2)} \sqrt{z+1}\, dS$$

using the surface from before.

$$\Phi(u,v) = (\overset{x}{\overset{\shortparallel}{2v\cos(u)}}, \overset{y}{\overset{\shortparallel}{2v\sin(u)}}, \overset{z}{\overset{\shortparallel}{v^2}})$$

$$f(x,y,z) = \sqrt{z+1}$$

$$f(\Phi(u,v)) = \sqrt{v^2+1}$$

$$\iint_{\Phi(B_2)} \sqrt{z+1}\ dS = \iint_{B_2} f(\Phi(u,v)) \underbrace{\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|}_{\text{as in \#2.}} du\,dv$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{v^2+1}\ 4v\sqrt{v^2+1}\ du\,dv = 2\pi \int_0^2 4v(v^2+1)\,dv$$

$$= 2\pi \int_0^2 4v^3 + 4v \ dv = 2\pi \left(v^4 + 2v^2 \right)_0^2 = 2\pi (16 + 8)$$

$$= 48\pi$$

4. Let A be the surface of the graph $z = x^2 - y^2$ over the unit box $[-1, 1] \times [-1, 1]$. Compute the integral

$$\iint_A \sqrt{x^2 + y^2 + 1/4}\, dS.$$

Either : if you're integrating over a graph

$$\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{4x^2 + 4y^2 + 1}$$

Or

$$\Phi(x,y) = (x, y, x^2 - y^2) \qquad \begin{array}{l} -1 \le x \le 1 \\ -1 \le y \le 1 \end{array}$$

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = (1, 0, 2x) \\[2ex] \dfrac{\partial \Phi}{\partial y} = (0, 1, -2y) \end{cases} \longrightarrow \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = (2x, -2y, -1)$$

$$= \nabla (x^2 - y^2 - z)$$

$$n = \| (2x, -2y, 1) \| = \sqrt{4x^2 + 4y^2 + 1} \qquad \not*$$

Scalar surface integral!

$$\iint_S f(x,y,z)\, \underbrace{dS} = \iint_D \overset{f(c(t))}{f(\Phi(u,v))}\, \underbrace{\overset{\|c'(t)\|}{\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|}\, du\, dv}$$

$$f = \sqrt{x^2+y^2+1/4} \qquad \Phi(x,y) = (x, y, x^2-y^2)$$

$$* \quad f(\Phi(x,y)) = \sqrt{x^2+y^2+1/4}$$

$$\iint_A \sqrt{x^2+y^2+\tfrac{1}{4}}\; dS = \int_{-1}^{1}\int_{-1}^{1} \sqrt{x^2+y^2+\tfrac{1}{4}}\, \sqrt{4x^2+4y^2+1}\; dx\, dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} \sqrt{x^2+y^2+\tfrac{1}{4}}\; 2\sqrt{x^2+y^2+\tfrac{1}{4}}\; dx\, dy$$

$$= 2\int_{-1}^{1}\int_{-1}^{1} x^2 + y^2 + \frac{1}{4}\,dx\,dy$$

$$= 2\int_{-1}^{1}\int_{-1}^{1} x^2 + y^2\,dx\,dy + 2\int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}\,dA$$

$$2\cdot\frac{1}{4}\cdot \text{Area}$$

$$= 2\cdot\frac{1}{4}\cdot 2\cdot 2$$

$$= 2$$

$$2$$

$$= 2\int_{-1}^{1}\left(\frac{1}{3}x^3 + xy^2\right)\Big|_{-1}^{1}\,dy$$

$$= 2\int_{-1}^{1}\left(\frac{1}{3} + y^2\right) - \left(-\frac{1}{3} - y^2\right)dy$$

$$= 2\int_{-1}^{1}\frac{2}{3} + 2y^2\,dy$$

$$= 4\int_{-1}^{1}\frac{1}{3} + y^2\,dy = 4\left(\frac{y}{3} + \frac{1}{3}y^3\right)\Big|_{-1}^{1}$$

$$= 8\left(\frac{1}{3} + \frac{1}{3}\right) + 2$$

$$= \frac{16}{3} + 2 = \boxed{\frac{22}{3}}$$