General Stuff

- Office Hours
  - T: 12:30 - 1:30, Th: 10 - 11
- Final Exam May 6th from 12:00pm - 3:00pm
- Quiz 6 Thursday 4/22
- Topics include ~~7.6 - 8.2~~ 7.6, 8.2 vector surface integrals Stokes' thm

  1 problem~~s~~

  15 minutes to take quiz

  5 minutes to upload to gradescope

  11:15 - 11:45 questions before quiz

  11:45 - 12:00 quiz

  12:00 - 12:05 uploading
- Lab 11 due tonight!

Plain Int:

1D

$$\int_a^b f(x)\,dx$$

2D

$$\iint_W f(x,y)\,dx\,dy$$

or

$dy\,dx$

3D

$$\iiint_W f(x,y,z)\,dV$$

$dx\,dy\,dz$

⋮

$dz\,dy\,dx$

Scalar

$$\int_c f\,ds$$

" $\|c'(t)\|$ "

$$\iint_S f(x,y,z)\,dS$$

" $\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\|$ "

Vector

$$\int_c F \cdot d\vec{s}$$

" $F \cdot c'(t)\,dt$ "

$$\boxed{\iint_S F \cdot d\vec{S}}$$

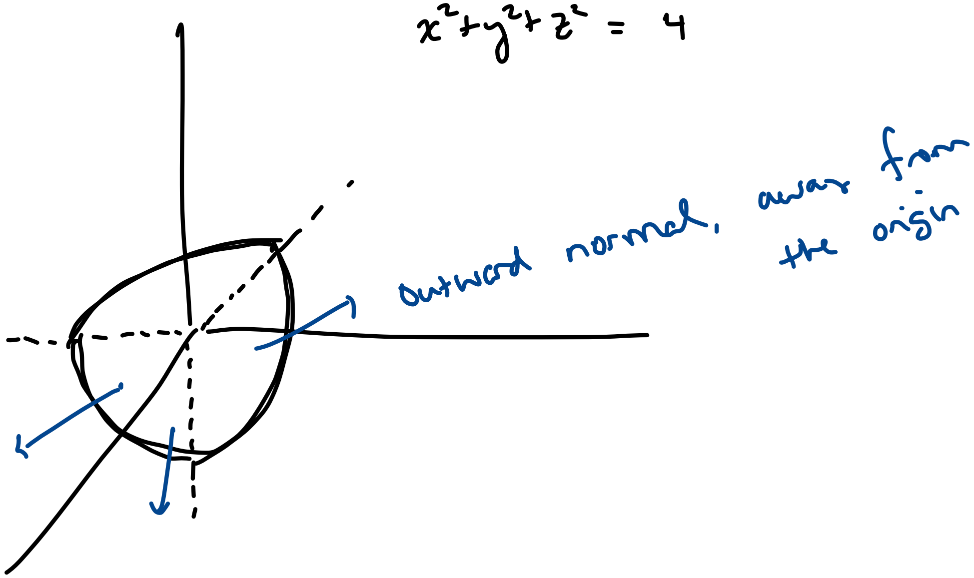
Today

$$= \iint F(\Phi(u,v)) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right)$$

$du\,dv$

1. Let $F = (0, 0, x^2)$. Calculate the flux integral of F through the surface given by the sphere of radius 2 such that $x, y, z \leq 0$, with outward facing normal.

$x^2+y^2+z^2 = 4$



outward!

$$\iint_{Sph} (0,0,x^2) \cdot d\vec{S} = \iint F(\Phi(\theta,\phi)) \cdot \left(\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \phi} \right) d\theta\, d\phi$$

What is the parametrization of Sph? Spherical coordinates!

$$\Phi(\theta,\phi) = (\overset{x}{2\cos\theta\sin\phi},\ \overset{y}{2\sin\theta\sin\phi},\ \overset{z}{2\cos\phi}) \qquad \rho = 2$$

$$\begin{array}{cccc} & i & j & k \\ \dfrac{\partial \Phi}{\partial \theta} = & \big(-2\sin\theta\sin\phi, & 2\cos\theta\sin\phi, & 0 \big) \end{array}$$

$$\frac{\partial \Phi}{\partial \phi} = \left(2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi \right)$$

$$n = \frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial \phi} = \Big(-4\cos\theta\sin^2\phi,\ -4\sin\theta\sin^2\phi,\ -4\overset{1}{\cancel{\sin^2}}\theta\cos\phi\sin\phi - 4\cancel{\cos^2}\theta\cos\phi\sin\phi \Big)$$

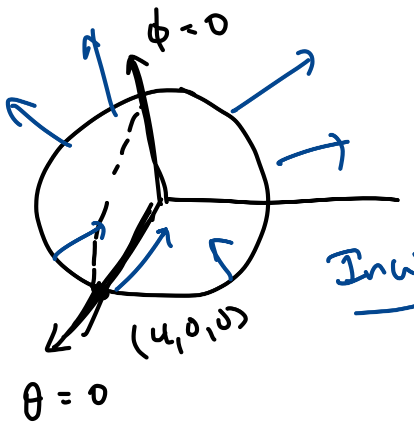
$$= \underline{\left(-4\cos\theta\sin^2\phi,\ -4\sin\theta\sin^2\phi,\ -4\cos\phi\sin\phi \right)}$$

Is this outward?

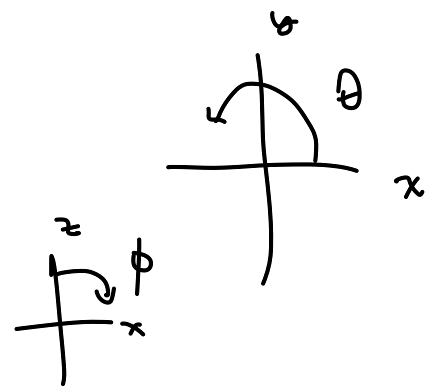
Inward!

$\theta = 0, \quad \phi = \pi/2$

$n(0, \pi/2) = \underline{(-4, 0, 0)}$



Inward!



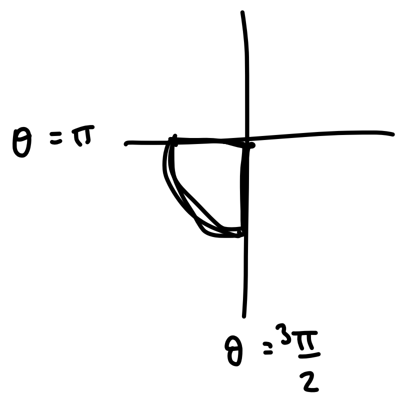
Outward $-n = \left(4\cos\theta \sin^2\phi,\ 4\sin\theta\sin^2\phi,\ 4\cos\phi\sin\phi \right)$

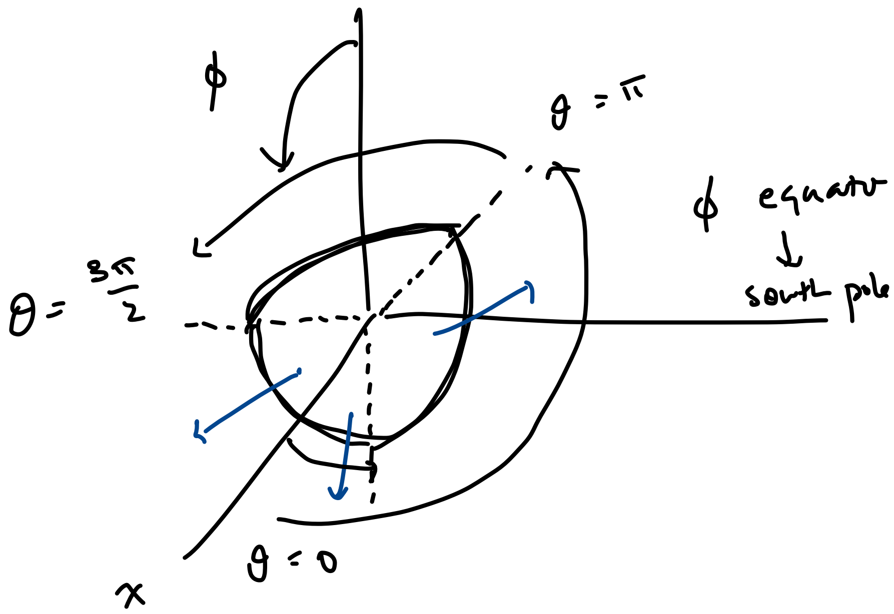
$F = (0,0,x^2)$

$F(\Phi(\theta,\phi)) = \left(0,\ 0,\ (2\cos\theta\sin\phi)^2\right) = \left(0,0,\ 4\cos^2\theta\sin^2\phi\right)$

$$\iint_{Sph} F(\Phi(\theta,\phi)) \cdot -\left(\frac{\partial\Phi}{\partial\theta} \times \frac{\partial\Phi}{\partial\phi} \right) d\theta\, d\phi$$

$$\int_{\frac{\pi}{2}}^{\pi} \int_{\pi}^{\frac{3\pi}{2}} \left(0,0,\ 4\cos^2\theta\sin^2\phi\right) \cdot \left(4\cos\theta \cancel{\sin}^2\phi,\ 4\sin\theta\cancel{\sin}^2\phi,\ 4\cos\phi\sin\phi \right) d\theta\, d\phi$$





$$= \int_{\frac{\pi}{2}}^{\pi} \int_{\pi}^{\frac{3\pi}{2}} 16\cos^2\theta \cos\phi \sin^3\phi \, d\theta \, d\phi = 16 \int_{\frac{\pi}{2}}^{\pi} \int_{\pi}^{\frac{3\pi}{2}} \cos^2\theta \cos\phi \sin^3\phi \, d\theta \, d\phi$$

$$= 16 \int_{\frac{\pi}{2}}^{\pi} \cos\phi \sin^3\phi \underbrace{\left(\int_{\pi}^{\frac{3\pi}{2}} \cos^2\theta \, d\theta \right)}_{\text{constant w/r/t}} d\phi$$

$$= 16 \left(\int_{\pi}^{\frac{3\pi}{2}} \cos^2\theta \, d\theta \right) \left(\int_{\frac{\pi}{2}}^{\pi} \cos\phi \sin^3\phi \, d\phi \right)$$

double angle

$u = \sin\phi \quad du = \cos\phi \, d\phi$

$$= 16 \left(\frac{\pi}{4} \right) \left(\frac{1}{4} \sin^4\phi \right)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 16 \frac{\pi}{4} \cdot \left(0 - \frac{1}{4} \right) = -\pi$$

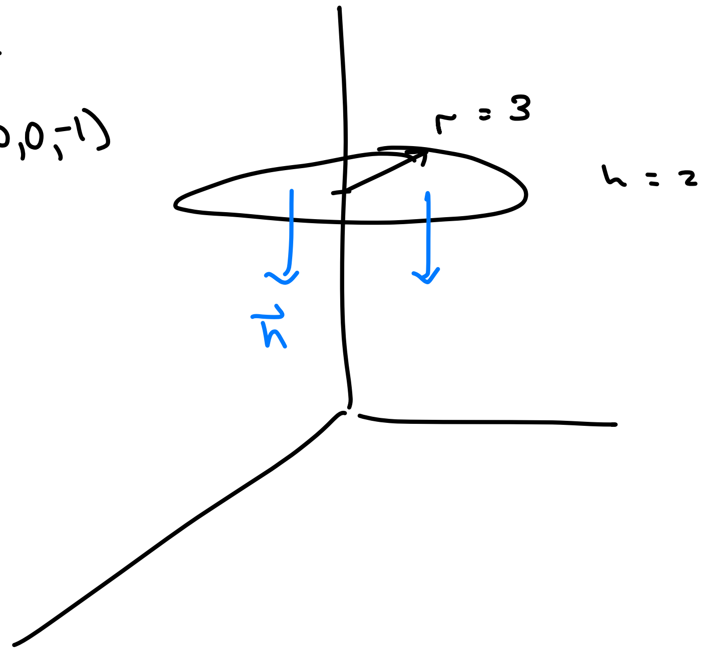
There's "$-\pi$" of force going outward

π force going inward

2. Let D be the disc of radius 3 at a height of 2. Let n be the downward facing normal. Compute the flux integral of $G = (x + y, x - y, z)$.

$$\iint_D G \cdot d\vec{S}$$

no matter what, the unit normal is $n = (0,0,-1)$



Two methods! Cartesian

$\Phi(x,y) = (x,y,2)$ $\quad -3 \le x \le 3 \quad -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} G(\Phi(x,y)) \left(\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} \right) dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x+y, x-y, 2) \cdot -(1,0,0) \times (0,1,0)\, dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x+y, x-y, 2) \cdot (0,0,-1)\, dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} -2\, dy\, dx$$

$= -2$ Area of disc $= -2(\pi (3)^2) = -18\pi$

<u>2nd Method!</u> Polar

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 2)$$

$$\frac{\partial \Phi}{\partial r} = (\cos\theta, \sin\theta, 0)$$

$$\frac{\partial \Phi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$n = (0, 0, r\cos^2\theta + r\sin^2\theta) = (0,0,r) \quad \text{upward}$$

$$(0,0,-r) \quad \text{downward!}$$

$$\iint_D G \cdot d\vec{S} = \int_0^{2\pi} \int_0^3 G(r\cos\theta, r\sin\theta, 2) \cdot (0, 0, -r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 (r\cos\theta + r\sin\theta, r\cos\theta - r\sin\theta, 2) \cdot (0, 0, -r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 -2r \, dr \, d\theta = -2(2\pi) \int_0^3 r \, dr$$

$$= -2(2\pi) \frac{1}{2} 3^2 = -18\pi$$

Same answer!

3. Let S be the surface given by the parametrization

$$\Phi(r, \theta) = (r\cos(\theta), r\sin(\theta), r^2)$$

from $r = 0$ to $r = 1$, and $\theta = 0$ to $\theta = 2\pi$. Compute the line integral around the counter-clockwise boundary ∂S of the vector field $F(x, y, z) = (x^2 - y, z^2 - x, x + y)$ using Stoke's Theorem.

$$\iint_S \nabla \times F \cdot d\vec{S} = \int_{\partial S} F \cdot d\vec{s}$$

which orientation does ∂S have?

CW or CCW?

vector surface

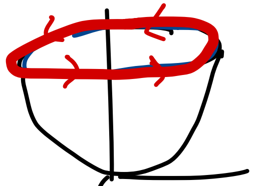
vector line

?? inward vs ?? outward

cw vs CCW

???

∂S



$$\int_{\partial S} (x^2 - y, z^2 - x, x + y) \cdot d\vec{s} = \underbrace{\iint_S \nabla \times (x^2 - y, z^2 - x, x + y) \cdot d\vec{S}}$$

∂S $(\cos\theta, \sin\theta, 1)$

$$\Phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r^2)$$

$$\nabla \times (x^2-y,\ z^2-x,\ x+y) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y & z^2-x & x+y \end{vmatrix}$$

$$= (1-2z,\ 0-1,\ -1-(-1)) = \underline{(1-2z, -1, 0)}$$

easier to integrate than F

$$= \iint_S (1-2z,\ -1,\ 0) \cdot d\vec{S}$$

$$\frac{\partial \Phi}{\partial \theta} = (-r\sin\theta,\ r\cos\theta,\ 0)$$

$$\frac{\partial \Phi}{\partial r} = (\cos\theta,\ \sin\theta,\ 2r)$$

$$\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial r} = (2r^2\cos\theta,\ 2r^2\sin\theta,\ -r\sin^2\theta - r\cos^2\theta)$$

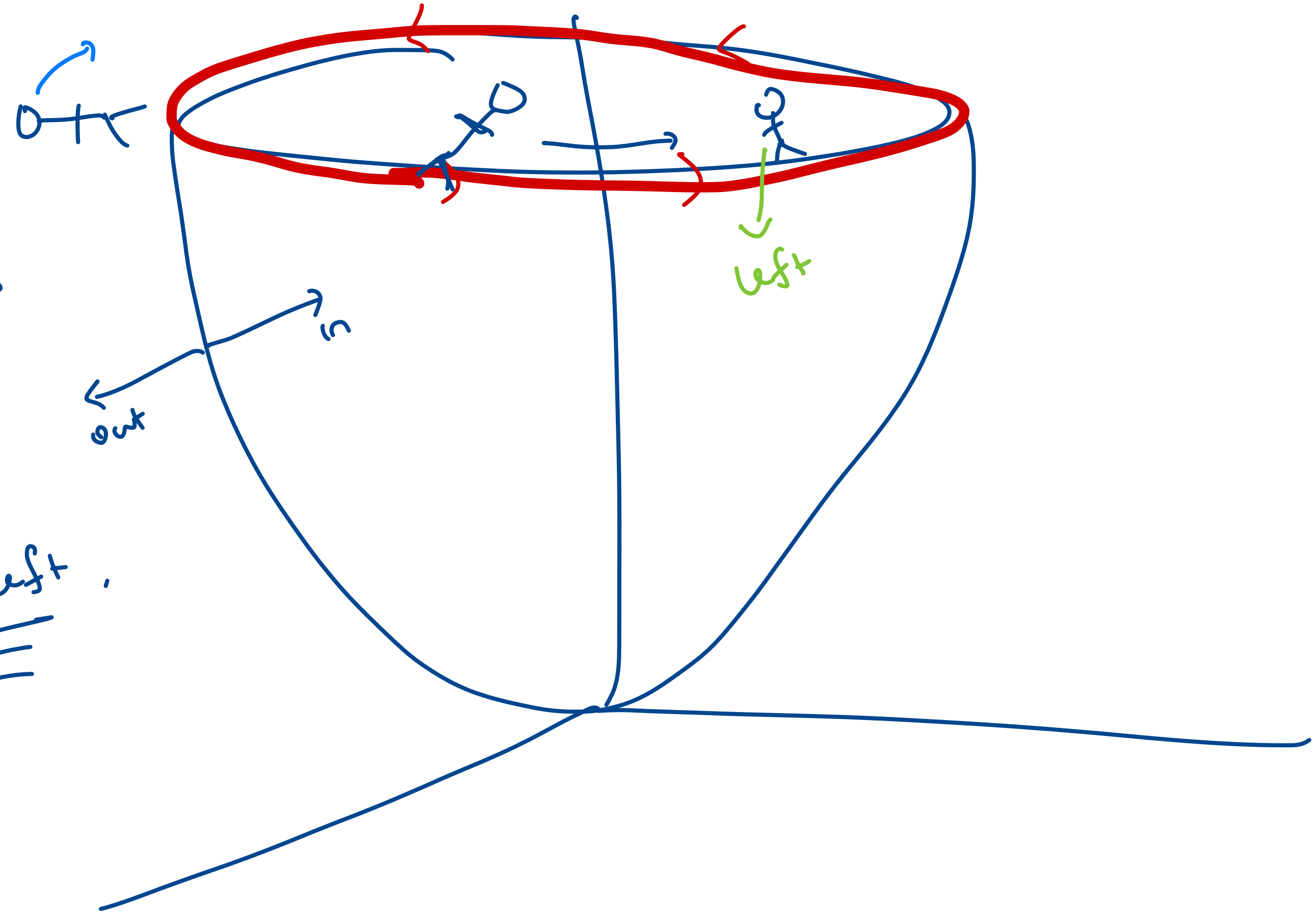
$$= (2r^2\cos\theta,\ 2r^2\sin\theta,\ -r)$$

CCW ∂S $\qquad$ Outward S ✗

OR

CW ∂S $\qquad$ Inward on S ✓

The orientations correspond when the person walks around the boundary w/ the surface on their left.



$$\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial r} = \left(2r^2\cos\theta,\ 2r^2\sin\theta,\ -r\sin^2\theta - r\cos^2\theta \right)$$

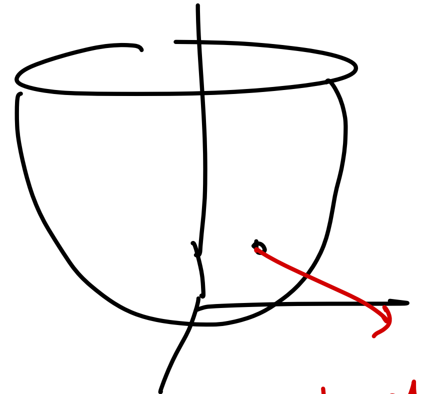
$$= \left(2r^2\cos\theta,\ 2r^2\sin\theta,\ -r \right)$$

Is this normal inward or outward?

✓ ✗

Pick a point! $(r,\theta) = (1,0)$

$$n(r=1,\ \theta=0) = (2\cdot 1,\ 2\cdot 0,\ -1) = (2,2,-1)$$



outward!

So we really want $-\left(\frac{\partial \Phi}{\partial \theta} \times \frac{\partial \Phi}{\partial r}\right)$

$$= \left(-2r^2\cos\theta,\ -2r^2\sin\theta,\ r \right)$$

$$\int_{\substack{\partial S \\ \text{ccw}}} F \cdot d\vec{s} = \iint_{\substack{S \\ \text{inward}}} \nabla \times F \cdot d\vec{S}$$

$$\int_{\substack{\partial S \\ \text{ccw}}} (x^2 - y,\ z^2 - x,\ x + y) \cdot d\vec{S} = \iint_{\substack{S \\ \text{inward}}} (1 - 2z,\ -1,\ 0) \cdot d\vec{S}$$

inward $\quad n = (-2r^2\cos\theta,\ -2r^2\sin\theta,\ r)$ $\qquad \Phi = (\underset{x}{r\cos\theta},\ \underset{y}{r\sin\theta},\ \underset{z}{r^2})$

$$= \int_0^{2\pi}\int_0^1 (1 - 2r^2,\ -1,\ 0) \cdot (-2r^2\cos\theta,\ -2r^2\sin\theta,\ r)\ dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1 - 2r^2)(-2r^2\cos\theta) + 2r^2\sin\theta \quad dr\, d\theta$$

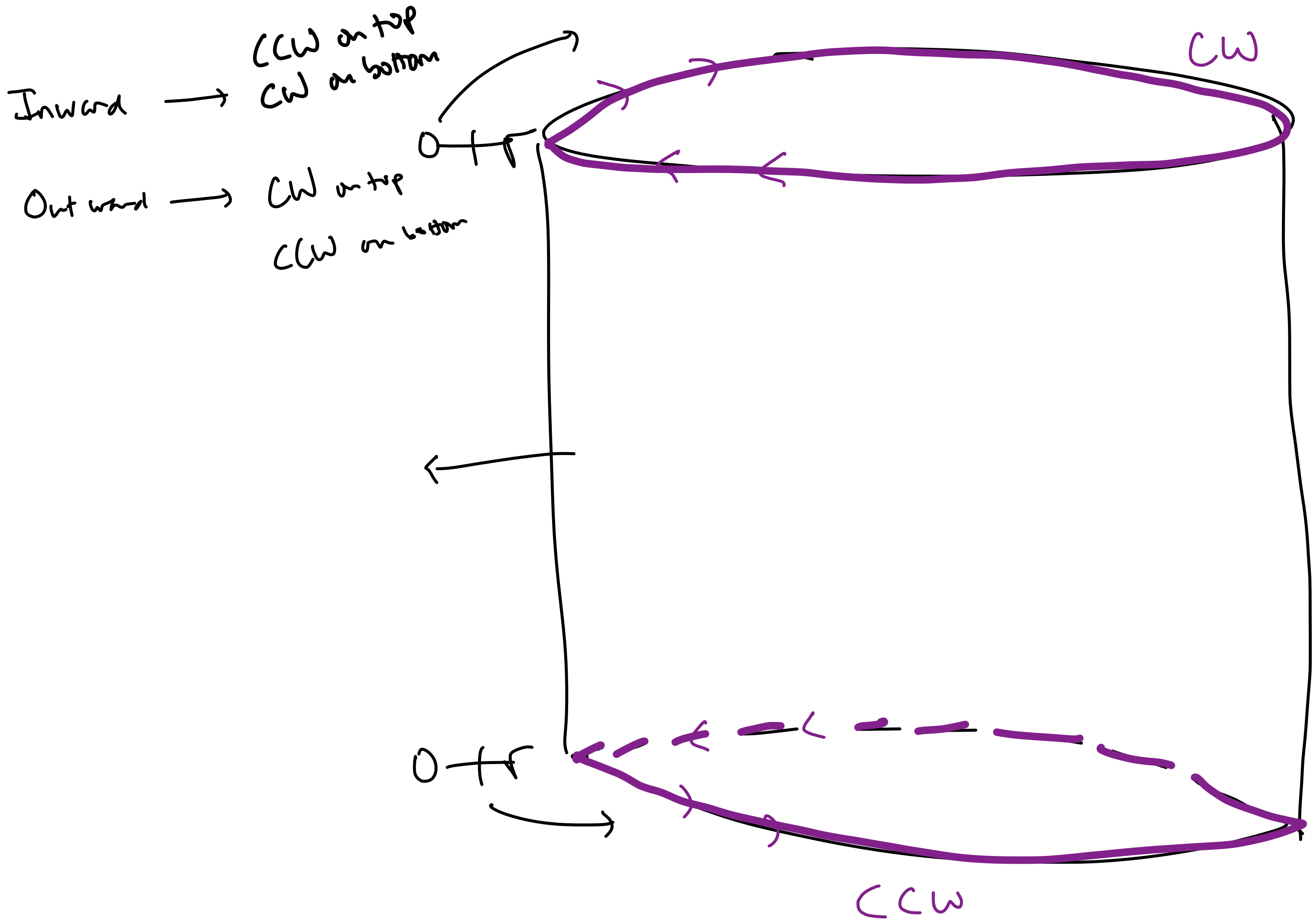
$$= \int_0^1\int_0^{2\pi} -2r^2\cos\theta + 4r^4\cos\theta + 2r^2\sin\theta \ dr\, d\theta$$

$$= \int_0^1 \left(-2r^2 \sin\theta + 4r^4 \sin\theta - 2r^2 \cos\theta \right)_0^{2\pi} dr\, d\theta$$

$$= \int_0^1 0 \, dr = 0 \qquad !$$

4. Let Cyl be the surface given by the cylinder of height 4 from $z = -2$ to $z = 2$ and radius $r = 3$. Let $G(x, y, z) = (x^2 + y^2, 0, z)$. Compute the integral $\hat{n}=$ outward

$$\iint_{\text{Cyl}} \nabla \times G \cdot dS.$$



Parametrization of bottom

CCW $\quad c_1(\theta) = (3\cos\theta, 3\sin\theta, -2)$

$G = (x^2+y^2, 0, z)$

Parametrization of top

CW $\quad c_2(\theta) = (3\cos\theta, -3\sin\theta, 2)$

$$\iint_{cyl} \nabla \times G = \int_{c_1 \text{ and } c_2} G \cdot d\vec{s} = \int_{\substack{c_1 \\ \text{CCW}}} G \cdot d\vec{s} + \int_{\substack{c_2 \\ \text{CW}}} G \cdot d\vec{s}$$

$$= \int_0^{2\pi} G(c_1) \cdot c_1'(\theta)\, d\theta + \int_0^{2\pi} G(c_2) \cdot c_2'(\theta)\, d\theta$$

$$= \int_0^{2\pi} (\overset{9}{\cancel{9\cos^2\theta + 9\sin^2\theta}}, 0, -2) \cdot (-3\sin\theta, 3\cos\theta, 0)\, d\theta$$

$$+ \int_0^{2\pi} (\overset{9}{\cancel{9\cos^2\theta + 9\sin^2\theta}}, 0, 2) \cdot (-3\sin\theta, -3\cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} -27 \sin\theta \, d\theta + \int_0^{2\pi} -27 \sin\theta \, d\theta$$

$$= -54 \int_0^{2\pi} \sin\theta \, d\theta = 0$$