General Stuff

no quiz or midterm on Thursday!

- Office Hours

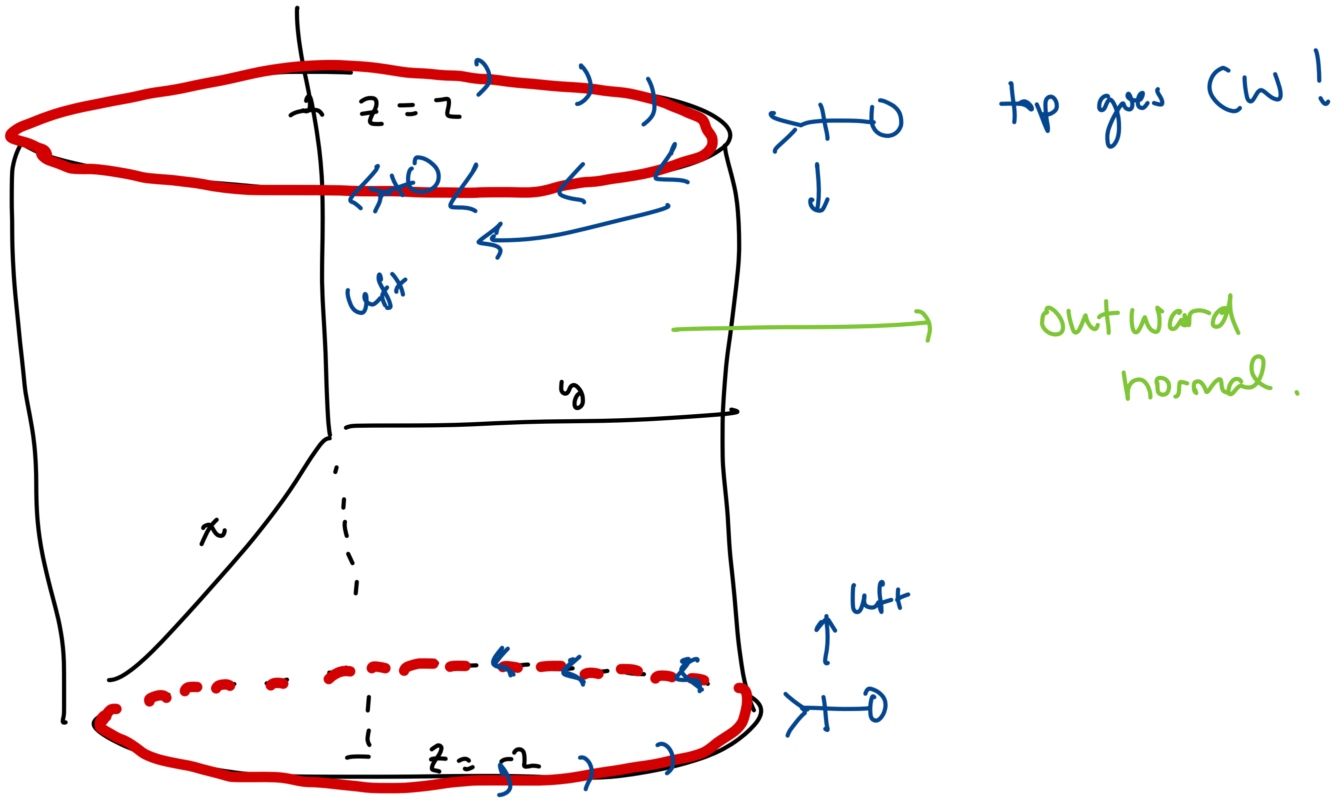
  T: 12:30 - 1:30, Th: 10 - 11 ✓
- Final Exam May 6th from 12:00pm - 3:00pm ✓
- Final Exam week office hours: Tues and Wed : 12 noon - 2pm in this Zoom room!
- Announcement: Lab 12 is the last lab, and we will only count your best 9 labs. I don't know what is going to happen in lab this week yet!

1. Let Cyl be the surface given by the cylinder of height 4 from $z = -2$ to $z = 2$ and radius $r = 3$. Let $G(x, y, z) = (x^2, y, z)$. Describe how Stokes' theorem applies to the integral

$$\iint_{\text{Cyl}} \nabla \times G \cdot dS.$$

Similar to problem 2 from Lab 12

Hollow cylinder!



2c)

$$\iint_{cyl} \nabla \times G \cdot d\vec{S} = \int_{\partial cyl} G \cdot d\vec{s} = \int_{top} G \cdot d\vec{s} + \int_{bottom} G \cdot d\vec{s}$$

CW
CCW ?

CW
CCW ?

$$= \int_{\text{top CW}} G \cdot d\vec{s} + \int_{\text{bottom CCW}} G \cdot d\vec{s}$$

$$= - \int_{\text{top CCW}} G \cdot d\vec{s} + \int_{\text{bottom CCW}} G \cdot d\vec{s}$$

2c) = − 2a) + 2b)

filled in sphere   hollow sphere

2. Let S be the sphere of radius 1 with surface denoted ∂S. Suppose ∂S has outward normal and let $F = (2x + y^2, 3y - \cos x, e^{xy} - z)$. Compute the integral

$$\iint_{\partial S} F \cdot dS.$$

W is a 3D region
∂W outward normal

$$\iint_{\partial W} F \cdot d\vec{S} = \iiint_W \nabla \cdot F \, dV$$

vector surface integral

plain old triple integral

parametrize ∂S $\rho \cos\phi \sin\theta$ etc....

$\frac{\partial \Phi}{\partial \phi} \times \frac{\partial \Phi}{\partial \theta}$, blah...

$$\iint_{\partial S} F \cdot d\vec{S} = \iint_{\partial S} (2x + y^2, 3y - \cos(x), e^{xy} - z) \cdot dS$$

$$= \iiint_{\text{sphere}} \operatorname{div}(F)\; dx\, dy\, dz$$

$$\nabla \cdot \left(2x+y^2,\ 3y-\cos(x),\ e^{xy}\cdot z\right) = \frac{\partial}{\partial x}\left(2x+\cancel{y^2}\right) + \frac{\partial}{\partial y}\left(3y - \cancel{\cos(x)}\right)$$

$$+ \frac{\partial}{\partial z}\left(\cancel{e^{xy}}\cdot z\right)$$

$$= 2+3-1 = 4\,!$$

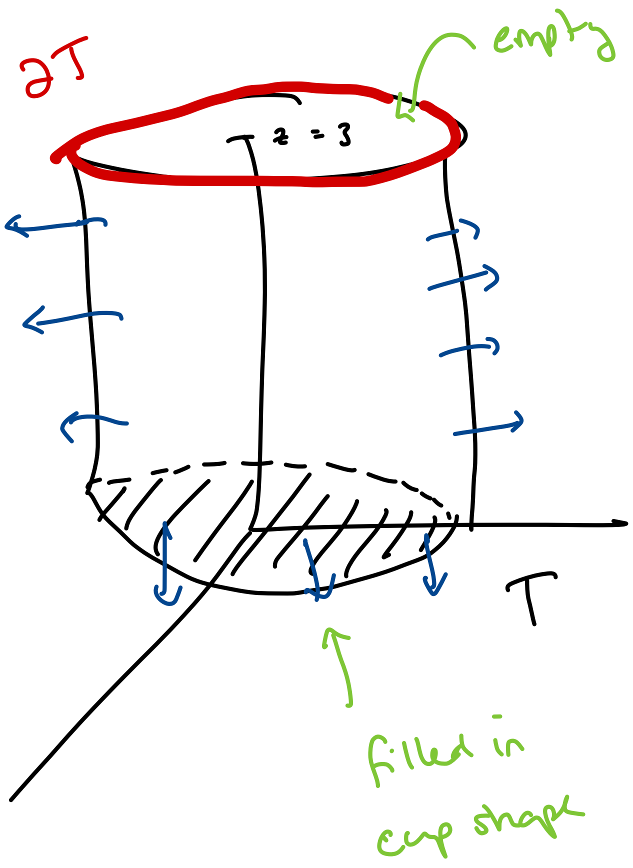
$$= \iiint_{\text{sphere}} 4 \; dV = \underbrace{4\iiint_{\text{sphere}} 1 \; dV}$$

$$= 4 \cdot \underbrace{\text{volume}(\text{sphere})} = 4\left(\frac{4}{3}\pi(1)^3\right) = \frac{16}{3}\pi \checkmark$$

See note!

3. Let T be the 2D surface defined by the cylinder $x^2 + y^2 = 4$ from $z = 0$ to $z = 3$ and the bottom hole of the cylinder is filled by the disc of radius 2 $(x^2 + y^2 = 4)$ in the xy-plane. Given T the outward normal. Compute the integral

$$\iint_T (x^2, y^2, z) \cdot dS.$$



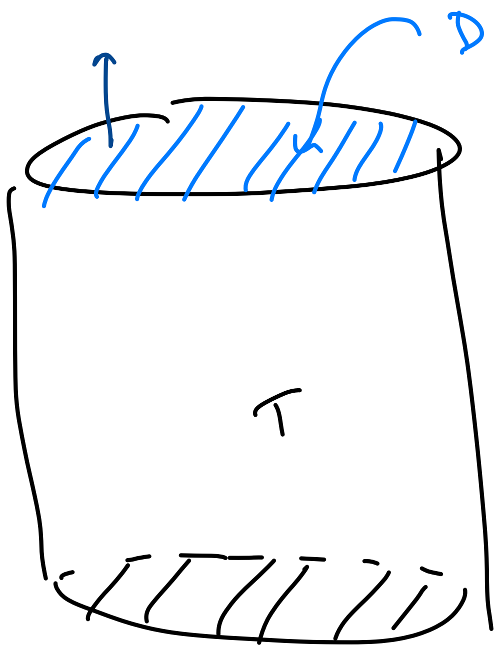
Note: T is not a closed surface!

You cannot apply Gauss Thm to T right now!

We can't really apply Stokes' Thm since that would require

$\nabla \times F = (x^2, y^2, z)$

Idea: Fill in the top to close the surface!



$T + D$ is a closed surface!

GT!

$$\iint_{T+D} (x^2, y^2, z) \cdot d\vec{S} = \boxed{\iint_T (x^2, y^2, z) \cdot dS} + \iint_D (x^2, y^2, z) \cdot d\vec{S}$$

solve!

$$= \iiint_{cyl} \operatorname{div}(x^2, y^2, z)\, dV$$

$$\operatorname{div}(x^2, y^2, z) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z)$$

$$= 2x + 2y + 1$$

$$= \iiint_{cyl} 2x + 2y + 1 \, dV$$

In total $\iiint_{cyl} 2x+2y+1 \, dV = \iint_T (x^2,y^2,z)\cdot d\vec{S} + \iint_D (x^2,y^2,z)\cdot d\vec{S}$

$$\iint_T (x^2,y^2,z)\cdot d\vec{S} = \underbrace{\overset{12\pi}{\iiint_{cyl} 2x+2y+1 \, dV}} - \underbrace{\iint_{\underline{D}} (x^2,y^2,z)\cdot d\vec{S}}$$

$h=3$ $r=2$ Jac

$$\iiint_{cyl} 2x+2y+1 \, dV = \int_0^3 \int_0^{2\pi} \int_0^2 (2r\cos\theta + 2r\sin\theta + 1) \, r \, dr \, d\theta \, dz$$

cartesian $\longrightarrow$ cylindrical

$$= 3 \int_0^{2\pi} \int_0^2 2r^2\cos\theta + 2r^2\sin\theta + r \, dr \, d\theta$$

$$= 3\int_0^2\int_0^{2\pi} 2r^2\cos\theta + 2r^2\sin\theta + r \; d\theta \, dr$$

$$= 3\int_0^2 \left(2r^2(\sin\theta) + 2r^2(-\cos\theta) + \theta r\right)_0^{2\pi} dr$$

$$= 3\int_0^2 \left(-\cancel{2r^2} + 2\pi r\right) - \left(-\cancel{2r^2}\right) dr$$

$$= 3\int_0^2 2\pi r \, dr = 6\pi\int_0^2 r\,dr = \frac{6\pi}{2}\left(r^2\right)_0^2$$

$$= 12\pi$$

$$\iint_D (x^2, y^2, z)\cdot d\vec{S}$$

$$\Phi(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 3)$$

D disc of rad 2 at height $z = 3$

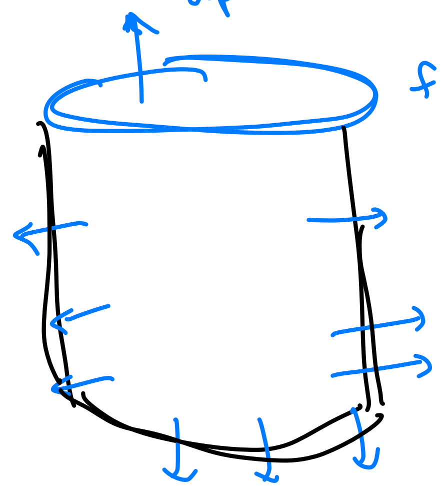
$r : 0 \to 2$

$\theta : 0 \to 2\pi$

$$\frac{\partial \Phi}{\partial r} = (\cos\theta, \sin\theta, 0)$$

$$\times$$

$$\frac{\partial \Phi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$



$$n = (0, 0, r\cos^2\theta + r\sin^2\theta) = \underline{(0,0,r)}$$

upward since $r > 0$

not a change of vars

$$\iint_{\underline{D}} (x^2, y^2, z) \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 (r^2\cos^2\theta, r^2\sin^2\theta, 3) \cdot (0, 0, r)\, dr\, d\theta$$

no Jacobian!

These were the variables to begin with!

$$= \int_0^{2\pi}\int_0^2 3r \, dr\, d\theta$$

$$= 6\pi \int_0^2 r\, dr = 6\pi \left(\tfrac{1}{2}r^2\right)_0^2 = 12\pi \quad \text{also!}$$

$$\iint_T (x^2, y^2, z) \cdot d\vec{S} \overset{!}{=} \underbrace{\overset{12\pi}{\iiint_{cyl} 2x + 2y + 1 \, dV}} - \underbrace{\overset{12\pi}{\iint_{D} (x^2, y^2, z) \cdot d\vec{S}}}$$

$$= 12\pi - 12\pi = 0$$

4. Let P be the parallelogram formed by the vectors $(1,1,-1),(1,-1,1)$, and $(-1,1,1)$. Suppose ∂P has inward normal. Evaluate the surface integral

$$\iint_{\partial P}(x+y^3,y-x^3,z+2)\cdot dS.$$

$$\iint_{\partial P}(x+y^3,\ y-x^3,\ z+2)\cdot dS = \iiint_P \nabla\cdot(x+y^3,\ y-x^3,\ z+2)\,dV$$

$$= \iiint_P \frac{\partial}{\partial x}(x+y^3) + \frac{\partial}{\partial y}(y-x^3) + \frac{\partial}{\partial z}(z+2)\,dV$$

$$= \iiint_P 1+1+1\,dV = 3\iiint_P 1\,dV$$

$$= 3\,\mathrm{vol}(P) = 3\left|\det\begin{bmatrix}1 & 1 & -1\\ 1 & -1 & 1\\ -1 & 1 & 1\end{bmatrix}\right| = 3\cdot 4 = 12$$

Note from #2

$$\iiint 1 \, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} 1 \quad J \, d\rho \, d\theta \, d\phi$$

cartesian $\longrightarrow$ spherical

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} 1 \quad \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

etc ...