General Stuff

- Office Hours

  Happening: Wednesday 5/5 from 12 noon - 2pm

  Maybe Happening: Tuesday 5/4 from 12 noon - 2pm *

  TAs may be coordinating office hours, so more info to come *
- Final Exam May 6th from 12:00pm - 3:00pm
- Announcement: Lab 12 is the last lab, and we will only count your best 9 labs.
- Today for lab period, we will be doing review!
- Please fill out my SRT (evals) when you get a chance. You can access it through through canvas or have received an email invitation.
- If you have DRC accomodations please let me know ASAP! *
- Shout out to you all for making it through the semester!

1. Consider the function $f(x, y) = \sin(xy) + e^{x+y}$. a) Find the Hessian matrix for f. b) Compute the second order Taylor polynomial for f expanded at the origin.

a) Recall the Hessian matrix $H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \boxed{\frac{\partial^2 f}{\partial y \partial x}} \\ \boxed{\frac{\partial^2 f}{\partial x \partial y}} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$

These are equal most of the time!

$f(x,y) = \sin(xy) + e^{x+y} = e^x e^y$

$\frac{\partial f}{\partial x} = y\cos(xy) + e^{x+y}$

$\frac{\partial f}{\partial y} = x\cos(xy) + e^{x+y}$

$\frac{\partial^2 f}{\partial x^2} = y^2(-\sin(xy)) + e^{x+y}$

$\frac{\partial^2 f}{\partial y \partial x} = 1\cos(xy) + y(-x\sin(xy)) + e^{x+y}$

$\|$

$\frac{\partial^2 f}{\partial x \partial y} = 1\cos(xy) + x(y(-\sin(xy)) + e^{x+y}$

$\frac{\partial^2 f}{\partial y^2} = x^2(-\sin(xy)) + e^{x+y}$

$$H = \begin{bmatrix} -y^2\sin(xy) + e^{x+y} & \cos(xy) - xy\sin(xy) + e^{x+y} \\ \cos(xy) - xy\sin(xy) + e^{x+y} & -x^2\sin(xy) + e^{x+y} \end{bmatrix}$$

For example

H at $(0,0)$

$$H(0,0) = \begin{bmatrix} -0^2\sin(0\cdot 0) + e^{0+0} & \cos(0) - 0 + e^0 \\ \cos(0) - 0 + e^0 & -0^2\sin(0) + e^{0+0} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

concavities in different directions

1. Consider the function $f(x,y) = \sin(xy) + e^{x+y}$. a) Find the Hessian matrix for f. b) Compute the second order Taylor polynomial for f expanded at the origin.

$$f(x,y) \approx \underbrace{\overbrace{f(0,0)}^{1} + \overbrace{\frac{\partial f}{\partial x}(0,0)}^{1}(x-0) + \overbrace{\frac{\partial f}{\partial y}(0,0)}^{1}(y-0)}_{1^{st}\text{ order}}$$

tangent plane

$$+ \underbrace{(x-0 \quad y-0)\,\overbrace{H(0,0)}\begin{pmatrix} x-0 \\ y-0 \end{pmatrix}}_{2^{nd}\text{ order}}$$

new!

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$f(0,0)$

$= \sin(0\cdot 0) + e^{0+0}$

$= 1$

$$\frac{\partial f}{\partial x} = y\cos(xy) + e^{x+y}$$

$$\frac{\partial f}{\partial x}(0,0) = (0\cdot\cos(0) + e^{0+0})$$

$$= 1$$

$$\frac{\partial f}{\partial y} = x\cos(xy) + e^{x+y}$$

$$\frac{\partial f}{\partial y}(0,0) = 0\cdot\cos(0) + e^{0+0} = 1$$

$$f(x,y) \approx \overbrace{f(0,0)}^{1} + \begin{pmatrix} x-0 & y-0 \end{pmatrix} \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$$

$$+ \begin{pmatrix} x-0 & y-0 \end{pmatrix} H(0,0) \begin{pmatrix} x-0 \\ y-0 \end{pmatrix}$$

$$\simeq 1 + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}$$

$$= 1 + (1x + 1y) + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1x + 2y \\ 2x + 1y \end{pmatrix}$$

$$= 1 + x + y + x(x+2y) + y(2x+y)$$

$$= 1 + x + y + x^2 + 4xy + y^2 \approx \sin(xy) + e^{x+y}$$

$$@ \ (0,0) \ !$$

2. Let $g(x,y) = 1 + x + 2y - x^2 - 2xy + y^2$. Re-center $g(x,y)$ from the origin to the point $(a,b) = (1,2)$.

$$g(x,y) = 1 + x + 2y - x^2 - 2xy + y^2 = 1 + \begin{pmatrix} x-0 & y-0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$+ \begin{pmatrix} x-0 & y-0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x-0 \\ y-0 \end{pmatrix}$$

$$g(x,y) = \underbrace{g(1,2)}_{5} + \begin{pmatrix} x-1 & y-2 \end{pmatrix} \begin{pmatrix} \overbrace{\frac{\partial g}{\partial x}(1,2)}^{-5} \\ \underbrace{\frac{\partial g}{\partial y}(1,2)}_{4} \end{pmatrix} + \begin{pmatrix} x-1 & y-2 \end{pmatrix} \underbrace{H(1,2)}_{\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

expansion @ $(1,2)$

$$g(1,2) = 1 + 1 + 2\cdot 2 - (1)^2 - 2(1)(2) + 2^2$$

$$= 2 + 4 - 1 - 4 + 4 = 5$$

$$\frac{\partial g}{\partial x} = 1 - 2x - 2y$$

$$\frac{\partial g}{\partial x}(1,2) = 1 - 2\cdot 1 - 2\cdot 2$$

$$= -5$$

$$g(x,y) = 1 + x + 2y - x^2 - 2xy + y^2$$

$$\frac{\partial g}{\partial y} = 2 - 2x + 2y \qquad \frac{\partial g}{\partial y}(1,2) = 2 - 2\cdot 1 + 2\cdot 2$$

$$= 4$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}(1 - 2x - 2y) = -2 \qquad \frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}(2 - 2x + 2y)$$

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial}{\partial y}(1 - 2x - 2y) = -2 = \frac{\partial^2 g}{\partial x \partial y} \qquad = 2$$

$$H = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$g(x,y) = \underbrace{g(1,2)}_{5} + (x-1 \quad y-2)\begin{pmatrix} \overbrace{\frac{\partial g}{\partial x}(1,2)}^{-5} \\ \underbrace{\frac{\partial g}{\partial y}(1,2)}_{4} \end{pmatrix} + (x-1 \quad y-2)\underbrace{H(1,2)}_{\begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}}\begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

$$\simeq 5 + \begin{pmatrix} x-1 & y-2 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} x-1 & y-2 \end{pmatrix} \begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$$

$$= 5 + -5(x-1) + 4(y-2) + \begin{pmatrix} x-1 & y-2 \end{pmatrix} \begin{pmatrix} -2(x-1) - 2(y-2) \\ -2(x-1) + 2(y-2) \end{pmatrix}$$

$$= 5 + -5(x-1) + 4(y-2) + -2(x-1)^2 - 4(x-1)(y-2) + 2(y-2)^2$$

Expansion of g @ $(1,2)$ polynomial in terms of $x-1$ and $y-2$

$\ast$ new!

3. a) Find the critical points of the function $g(x, y) = 1 + x + 2y - x^2 - 2xy + y^2$. b) Use the Hessian matrix to determine whether the critical points are minima or maxima.

a) The critical points of a function are such that $\nabla f(x_0, y_0) = \vec{0}$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ system of equations!

$$g(x, y) = 1 + x + 2y - x^2 - 2xy + y^2$$

$$\nabla g = \begin{pmatrix} 1 - 2x - 2y \\ 2 - 2x + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$1 - 2x - 2y = 0$$

$$2 - 2x + 2y = 0$$

$$-2x - 2y = -1$$

$$-2x + 2y = -2$$

$$\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\nabla g\left(\frac{-3}{4}, \frac{1}{4}\right) = (0,0)$$

$$= \frac{1}{2\cdot(-2) - 2\cdot 2}\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{-1}{8}\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{-1}{8}\begin{pmatrix} -6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1/4 \end{pmatrix}$$

So $\left(\frac{-3}{4}, \frac{1}{4}\right)$ is the only critical point!

Does a min or max occur here?

Min or max is controlled by $H\left(\frac{-3}{4}, \frac{1}{4}\right)$!

$$H = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$$

@ every point! constant H.

$$H\left(\frac{-3}{4}, \frac{1}{4}\right) = \begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$$

min? $\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix} \overset{?}{>} 0$

max? $\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix} \overset{?}{<} 0$

To memorize

$$H = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (x_0, y_0)$$

$$\det \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix} = -2 \cdot -2 - (-2 \cdot 2) = -4 - (+4) = -8$$

- (x_0, y_0) is a local min if $a > 0$ and $\det H > 0$
- (x_0, y_0) is a local max if $a < 0$ and $\det H > 0$.

saddle point!

4. Find the local minima and maxima of the function $h(x, y) = \frac{1}{3}x^3 + xy^2 - x + y$.

$$\nabla h = 0 \implies \begin{cases} x^2 + y^2 - 1 = 0 \\ 2xy + 1 = 0 \end{cases} \longrightarrow y = \frac{-1}{2x}$$

$$x^2 + \left(\frac{-1}{2x}\right)^2 - 1 = 0$$

$$x^2 + \frac{1}{4x^2} - 1 = 0$$

$$4x^4 + 1 - 4x^2 = 0$$

$$4x^4 - 4x^2 + 1 = 0$$

$$(2x^2 - 1)^2 = 0$$

$$2x^2 - 1 = 0$$

$$x^2 = \frac{1}{2} \implies x = \pm\sqrt{\frac{1}{2}}$$

$$\longrightarrow y = \frac{-1}{2x}$$

$$y = \frac{-1}{2\left(\pm\sqrt{\frac{1}{2}}\right)} = \mp \frac{1}{2/\sqrt{2}} = \mp\sqrt{\frac{1}{2}}$$

So the two critical points are

$$\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) \text{ and }$$

$$\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right).$$

$H(h)$ ...

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(x^2 + y^2 - 1\right) = 2x$$

$$H = \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}$$

$$\frac{\partial^2 h}{\partial y\, dx} = \frac{\partial}{\partial y}\left(x^2 + y^2 - 1\right) = 2y = \frac{\partial^2 h}{\partial x \partial y}$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(2xy + 1\right) = 2x$$

$$H\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$$

$$H\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = \begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$\det H = 2 + 2 = 0$

$\det H = 0$

no info !! ⌒

Double Integrals

$$\iint_W f(x,y)\, dA$$

$dx\, dy$

or $dy\, dx$

Two single variable integrations

- Volume under the surface $f(x,y)$ above W

Triple Integral

$$\iiint_W f(x,y,z)\, dV$$

$dx\,dy\,z$ or 5 other orders

Three single variable integrations

" - weight sum of all function values in W "

$$\iiint_W 1\, dV = \mathrm{vol}(W)$$

Scalar line integrals

or arclength integral

or scalar path integral

$$\int_c f\, ds = \int_a^b f(c(t))\, \|c'(t)\|\, dt$$

parametrized curve

$$\int_c 1\, ds = \text{arclength of } c$$

Vector line integrals

$$\int_C F \cdot d\vec{s} = \int_a^b F(c(t)) \cdot c'(t)\,dt$$

parametrized curve

vector field

look for the dot product

vector quantity

Work done by F on a particle moving along C

Scalar surface integral

$$\iint_S f(x,y,z)\,dS = \underbrace{\iint_{D \subseteq \mathbb{R}^2} f(\Phi(u,v)) \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| du\,dv}_{\text{In total this a plain old double integral}}$$

parametrized surface

$\Phi(u,v) : D \to \mathbb{R}^3$

$$\iint_S 1\,dS = \text{surface area of } S$$

Vector Surface integral

$$\iint_S F \cdot d\vec{S} = \iint_D F(\Phi(u,v)) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right) du\,dv$$

parametrized surface

$\Phi(u,v) : D \to \mathbb{R}^3$

look for dot product

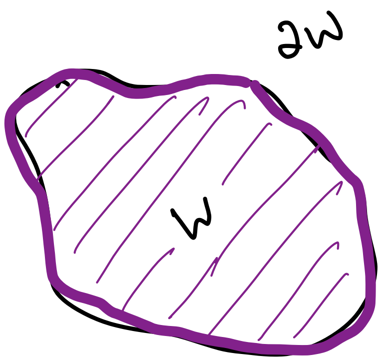
measures the flow of the vector field through S.

Green's theorem

vector line integral

plain old double integral

$$\int_{\partial W} (P,Q) \cdot d\vec{s} = \pm \iint_W \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$



+ ∂W CCW

− ∂W CW

- Either (P, Q) is too complicated but $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is easy
- W is complicated but ∂W is easy

Stokes' theorem

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \pm \iint_S \nabla \times F \cdot d\vec{S}$$

is the surface on the left or right

$+$ orientations of ∂S and S agree

$-$ not agree

- F is complicated but $\nabla \times F$ is simple.

- $\iint \nabla \times F \cdot dS = ??$
  I don't feel like taking the curl !!
  oh $= \int F \cdot d\vec{s}$ ✓

- It might be that $\nabla \times F = 0$

$$\nabla \times (\cos(x), \sin(y), z)$$

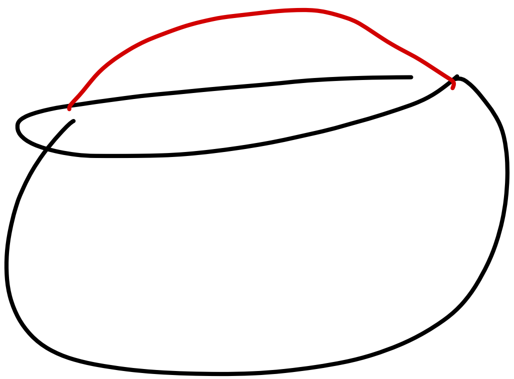
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & \sin y & z \end{vmatrix} = (0,0,0)$$

Gauss' Theorem

S has <u>no boundary</u>

$$\iint_S F \cdot dS = \underbrace{\iiint_{int(S)} div(F)\, dV}_{\text{plain old triple integral}}$$

- If S is complicated but $int(S)$ is easy
- If F is complicated but $div(F)$ easy
- Close the surface $\longrightarrow$ Gauss's theorem.



$$\iiint_{sphere} - \iint_{top}$$

5. Let P be the parallelogram formed by the vectors $(1,1,-1),(1,-1,1)$, and $(-1,1,1)$. Suppose ∂P has inward normal. Evaluate the surface integral

$$\iint_{\partial P} (x+y^3, y-x^3, z+2)\cdot dS.$$

$$\iint_{\partial P} (x+y^3,\ y-x^3,\ z+2)\cdot dS = \iiint_P \nabla\cdot(x+y^3,\ y-x^3,\ z+2)\, dV$$

$$= \iiint_P \frac{\partial}{\partial x}(x+y^3) + \frac{\partial}{\partial y}(y-x^3) + \frac{\partial}{\partial z}(z+2)\, dV$$

$$= \iiint_P 1+1+1\, dV = 3\iiint_P 1\, dV$$

$$= 3\,\mathrm{vol}(P) = 3\left|\det\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right| = 3\cdot 4 = 12$$

6. Find the total derivative of the function $F(x,y,z) = (xz + e^y, x + y^2 - \cos(z))$ at the point $(x_0, y_0, z_0) = (1,0,0)$.

$$\underset{\text{3 inputs}}{F : \mathbb{R}^3} \longrightarrow \underset{\text{2 outputs}}{\mathbb{R}^2}$$

SWAP!

DF 2×3 matrix

$$DF = \begin{array}{c|ccc} & x & y & z \\ \hline F_1 & & & \\ F_2 & & & \end{array}$$

$$DF = \begin{pmatrix} \frac{\partial}{\partial x}(xz + e^y) & \text{etc} \\ & - - - - \end{pmatrix} = \begin{pmatrix} z & e^y & x \\ 1 & 2y & \sin(z) \end{pmatrix}$$

$$DF(1,0,0) = \begin{pmatrix} 0 & e^0 & 1 \\ 1 & 2 \cdot 0 & \sin(0) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

7. Let $F(x,y,z) = (xz + e^y, x + y^2 - \cos(z))$ as before and $G(u,v) = (2u - v, uv, e^u)$. Calculate $D(G \circ F)(1,0,0)$ using the chain rule.

$$G(F) : \mathbb{R}^3 \xrightarrow[1]{F} \mathbb{R}^2 \xrightarrow[2]{G} \mathbb{R}^3$$

$G \circ F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$D(G \circ F)$ is 3×3

DF 2×3 matrix

DG 3×2 matrix

$$\underset{3\times 3}{D(G\circ F)(1,0,0)} = \underset{\boxed{3}\times \cancel{2}}{DG(F(1,0,0))}\ \underset{\cancel{2}\times\boxed{3}}{DF(1,0,0)} \longrightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$F(1,0,0) = (1\cdot 0 + e^0,\ 1 + 0^2 - \cos(0))$$

$$= (1,\ 1-1) = (1,0)$$

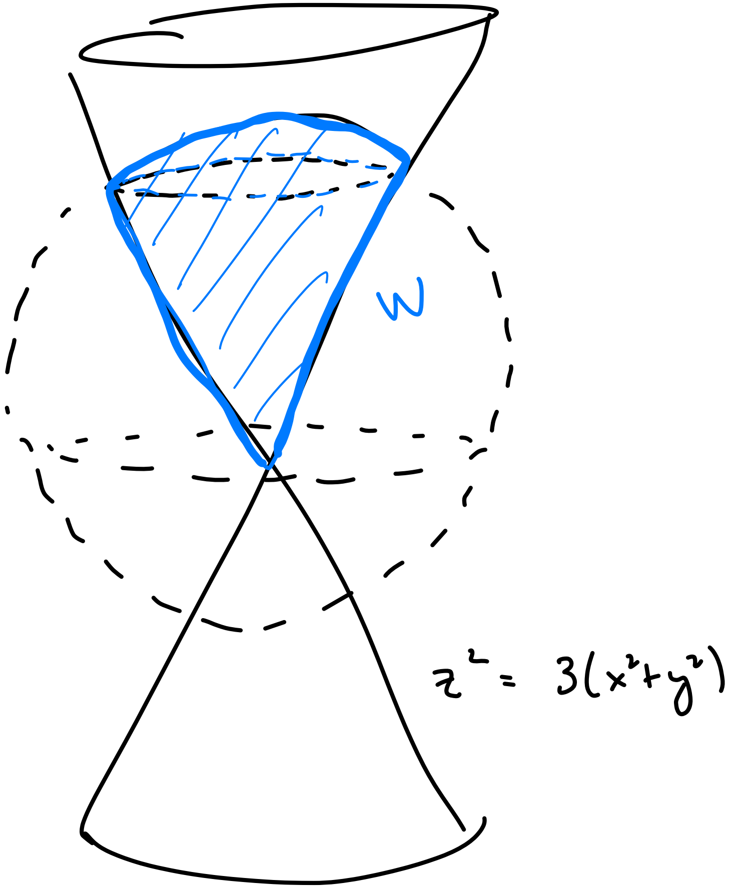
$$DG = \begin{array}{c} \\ G_1 \\ G_2 \\ G_3 \end{array}\begin{array}{cc} u & v \\ \multicolumn{2}{c}{\left[\begin{array}{cc} & \\ & \\ & \end{array}\right]} \end{array} = \begin{bmatrix} 2 & -1 \\ v & u \\ e^u & 0 \end{bmatrix}$$

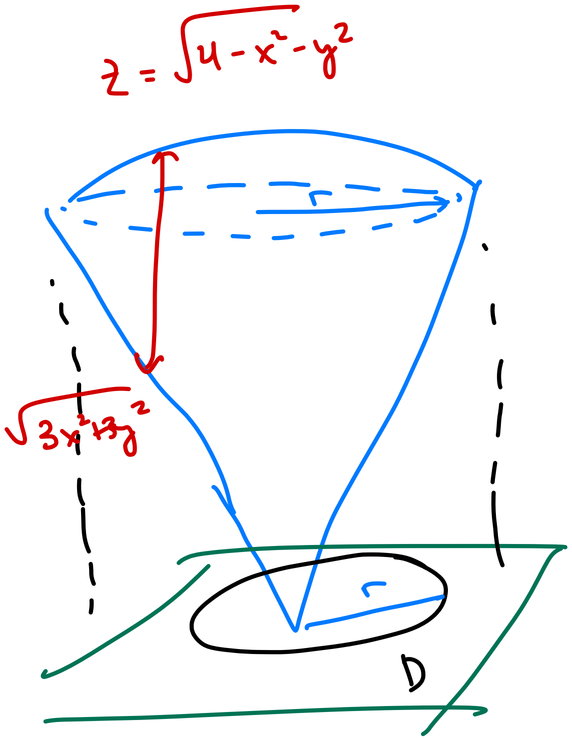
$$DG(1,0) = \begin{array}{cc} u & v \\ \multicolumn{2}{c}{\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ e & 0 \end{bmatrix}} \end{array}$$

$$D(G\circ F)(1,0,0) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ e & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \square & & \\ & & \square \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ e & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & e & e \end{bmatrix} \checkmark$$

8. Let W be the region bounded by $z^2 = 3x^2 + 3y^2$ and the sphere $x^2 + y^2 + z^2 = 4$. Evaluate the triple integral $\iiint_W x - 1\, dV$





This region is z-simple!

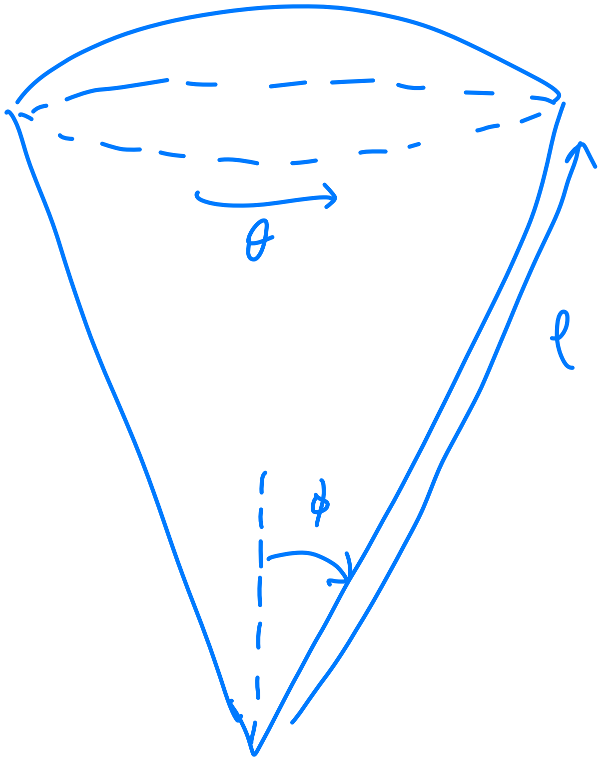
Shadow is a circle of some radius.

$x^2+y^2+(3x^2+3y^2) = 4$

$4x^2+4y^2 = 4 \implies r = 1$

$$\iiint_W x-1 \, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{4-x^2-y^2}} x-1 \, dz \, dy \, dx$$

Let's try another set of coordinates ---

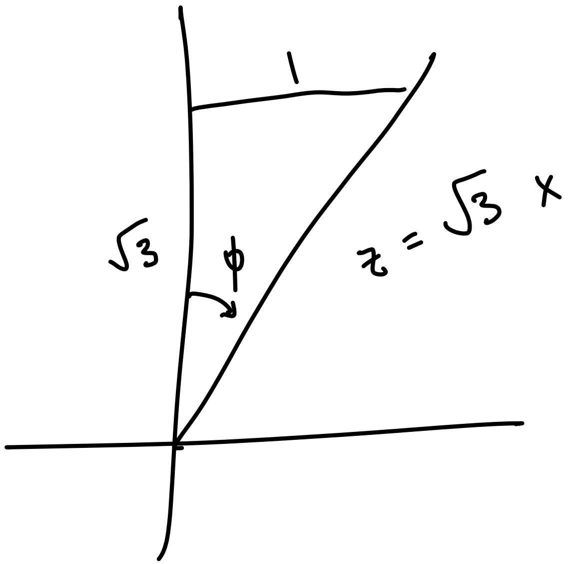


Since we are bounded by $x^2+y^2+z^2=4$

$\{\rho, 0, 2\}$.

$\{\theta, 0, 2\pi\}$ as usual.

when does ϕ stop at ?



$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$$

$$\{\phi, 0, \pi/6\}$$

$$x = \rho \cos\theta \sin\phi$$

$$J = \rho^2 \sin\phi$$

$$\Rightarrow \quad \iiint_W x - 1 \, dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 (\rho \cos\theta \sin\phi - 1)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^3 \cos\theta \sin^2\phi - \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

since $\int_0^{2\pi} \cos\theta \, d\theta = 0$

$$= \int_0^{2\pi} \int_0^{\pi/6} 4 \cancel{\cos\theta} \sin^2\phi - \frac{8}{3} \sin\phi \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\pi/6} -\frac{8}{3} \sin\phi \, d\phi$$

$$= -\frac{16\pi}{3} \Big(-\cos\phi \Big)_0^{\pi/6} = -\frac{16\pi}{3} \left(-\frac{\sqrt{3}}{2} + 1 \right)$$

$$= \boxed{\frac{16\pi}{3} \left(1 - \frac{\sqrt{3}}{2} \right)}$$

9. Let a particle p be moving along the trajectory $r(t) = (t, t-1, t^2-2)$ from $t = 0$ to $t = 2$. a) How far does the particle travel in those 2 seconds? b) Find the acceleration of the particle as a function of time. c) Find the total amount of work done by the $F = (x, y^2, -z)$ on p.

a)

arclength of r $= \int_r 1 \, ds = \int_0^2 \|r'(t)\| \, dt = \int_0^2 \|(1, 1, 2t)\| \, dt$

$= \int_0^2 \sqrt{1^2 + 1^2 + (2t)^2} \, dt = \int_0^2 \sqrt{2 + 4t^2} \, dt \approx 5.124$

$t = \frac{1}{\sqrt{2}} \tan\theta \qquad dt = \frac{1}{\sqrt{2}} \sec^2(\theta) \, d\theta$

$\int \sqrt{2 + 2\tan^2\theta} \, \frac{1}{\sqrt{2}} \sec^2\theta \, d\theta = \frac{1}{\sqrt{2}} \sqrt{2} \int \sec^3\theta \, d\theta$

ugh sorry...

b) $a(t) = r''(t) = \frac{d}{dt}(1, 1, 2t) = (0, 0, 2)$

c) $$\int_C (x, y^2, -z) \cdot ds = \int_0^2 (t, (t-1)^2, 2-t^2) \cdot (1, 1, 2t)\, dt$$

$$= \int_0^2 t + (t-1)^2 + 2t(2-t^2)\, dt$$

$$= \int_0^2 t + t^2 - 2t + 1 + 4t - 2t^3\, dt$$

$$= \int_0^2 -2t^3 + t^2 + 3t + 1\, dt = \frac{8}{3}$$

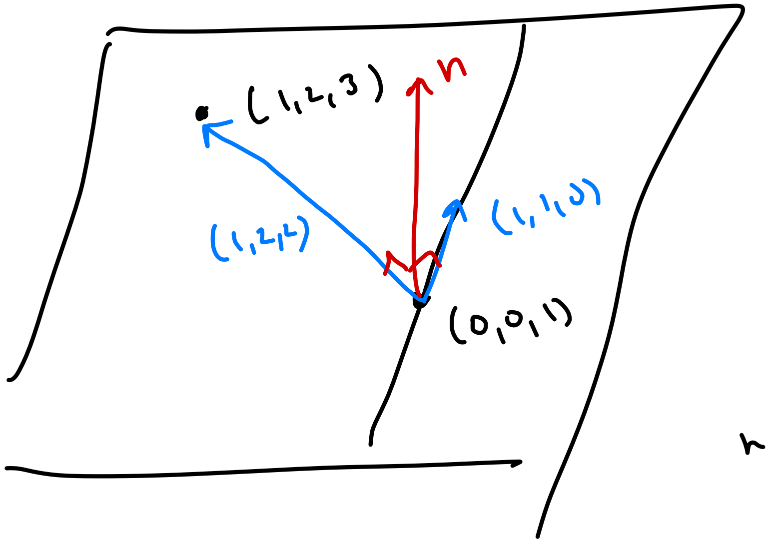
10. Find an equation for the plane which contains the point $(1,2,3)$ and the line $\ell(t) = (0,0,1)+(1,1,0)t$.

$$\ell(t) = \underbrace{(0,0,1)}_{\text{point on line}} + \underbrace{(1,1,0)}_{\text{direction vector}}t = (t,t,1)$$

direction vector for plane

plane

$$n \cdot (\vec{x} - \vec{a}) = 0$$



$$\vec{v}_2 = (1,2,3) - (0,0,1) = (1,2,2)$$

$$n = (1,2,2) \times (1,1,0)$$

$$= \begin{vmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (-2, 2, -1)$$

$$(-2, 2, 1) \cdot ((x, y, z) - (0, 0, 1)) = 0$$

$$-2(x) + 2(y) + 1(z-1) = 0$$

$$-2x + 2y + z = 1$$

11. a) Show that the vector field

$$F = (\sin(y) + 2, x\cos(y) + 1)$$

is conservative. b) Find a potential function $\phi(x, y)$ such that $\nabla\phi = F$. c) Evaluate the integral $\int_c F \cdot ds$ where $c(t) = (-1 + t^2, 4t^2 - 1)$ from $t = -1/2$ to $t = 1/2$.

a) $\nabla \times F = 0$ is the fastest way!

$$\nabla \times F = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(y)+2 & x\cos(y)+1 & 0 \end{vmatrix} \end{array} = \left(0, 0, \frac{\partial}{\partial x}(Q) - \frac{\partial}{\partial y}(P)\right)$$

$$= \cos y - \cos(y) = 0$$

It's conservative!

b) $\nabla\phi = (\sin(y)+2,\ x\cos(y)+1)$

$$\frac{\partial\phi}{\partial x} = \sin(y)+2 \longrightarrow \phi = x\sin(y) + 2x + f(y)$$

$$\frac{\partial\phi}{\partial y} = x\cos(y)+1 \longrightarrow \phi = x\sin(y) + y + g(x)$$

$$\Longrightarrow \quad \phi = x\sin(y) + 2x + y$$

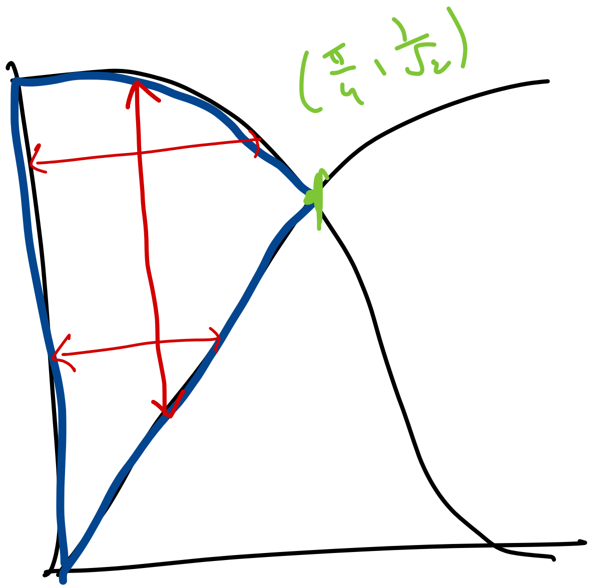
c) $$\int_C F\cdot ds = \int_C \nabla\phi \cdot ds = \phi(\text{end}) - \phi(\text{start}) = 0$$

end: $c\left(\frac{1}{2}\right) = \left(-1+\left(\frac{1}{2}\right)^2,\ 4\left(\frac{1}{2}\right)^2-1\right) = \left(\frac{-3}{4}, 0\right) \quad \phi\left(\frac{-3}{4}, 0\right) = \frac{-3}{2}$

$c\left(\frac{-1}{2}\right) = \left(-1+\left(\frac{-1}{2}\right)^2,\ 4\left(\frac{-1}{2}\right)-1\right) = \left(\frac{-3}{4}, 0\right)$ Closed curve!

$0 \leq x$

12. Let R be the region bounded between $x \leq 0$, $y = \cos(x)$, and $y = \sin(x)$. a) Determine whether the region is x-simple or y-simple. b) Find the area of R. Write out the integrals for both $dx\,dy$ and $dy\,dx$ but only solve one of them.



Not x-simple since the bounds change!

It's y-simple! Always $\sin(x) \longrightarrow \cos(x)$.

$\sin(x) = \cos(x)$

$\tan(x) = 1 \implies x = \pi/4$

$$\iint_R 1 \, dy\,dx = \int_0^{\pi/4} \int_{\sin(x)}^{\cos(x)} 1 \, dy\,dx = \int_0^{\pi/4} \cos(x) - \sin(x) \, dx$$

$$= \Big(\sin(x) + \cos(x)\Big)_0^{\pi/4} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (1) = \sqrt{2} - 1$$

The other integral would be

$$\int_0^{\frac{1}{\sqrt{2}}} \int_0^{\arcsin(y)} 1 \, dx \, dy + \int_{\frac{1}{\sqrt{2}}}^{1} \int_0^{\arccos(y)} 1 \, dy \, dx$$

bleh... :(