Review Problems for Midterm 1

Here is a list of review problems for midterm 1. This list is designed to help students understand the type of question which could be asked. This list is not exhaustive. Students are responsible for all of the material in the homework problems (especially the starred ones!), as well as all material from lecture and material in the text from the sections covered.

1. Solve the system
$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$$
by
   (a) Gaussian elimination.
   (b) first finding the LU factorization of A, then using forward and back substitution.
   (c) matrix inversion.

2. Calculate the (permuted if necessary) LDV factorization of the following matrix.
$$\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 0 & 4 & 1 \\ 1 & 3 & 6 & 4 \end{bmatrix}$$

3. Show that any symmetric, regular matrix A can be factored as $A = LDL^T$, where L is special lower triangular and D is diagonal. Hint - What factorization must A have as a regular matrix?

4. Indicate which of the following statements are True and which are False. Write a short explanation for or give a counter-example to the starred statements.
   ∗(a) If A is a nonsingular symmetric matrix, so is A^{-1}.
   ∗(b) If A and B are symmetric matrices, so is AB.
   (c) If A and B are upper triangular matrices, so is AB.
   (d) If A is a square $n \times n$ matrix and $\boldsymbol{b}$ is a vector in $\mathbb{R}^n$, then $A\boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\det A \neq 0$.
   (e) If A is a square $n \times n$ matrix and $\det A \neq 0$, then A^{-1} exists.
   (f) If A is a square $n \times n$ matrix, then $\det(A + B) = \det A + \det B$.
   ∗(g) If A and B are a square $n \times n$ matrices, then $\det(A \cdot B^{-1}) = \frac{\det A}{\det B}$.
   (h) If A is a square $n \times n$ matrix and $\det A = 0$, then A has rank smaller than n.
   ∗ (i) If V is a vector space and U and W are subspaces of V, then so is $U + W$.
   ∗ (j) If V is a vector space and U and W are subspaces of V, then so is $U \cap W$.
   ∗(k) If V is a vector space and U and W are subspaces of V, then so is $U \cup W$.
   (l) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If the dimension of the image of L is m, the dimension of the kernel of L is $n - m$.

5. For each of the following sets, determine if they form a vector space (over $\mathbb{R}$). If so, find a basis and record the dimension. If not, explain why not.

   (a) The set of all polynomials of odd degree no larger than 5.

   (b) The set of solutions to $y''(t) + y'(t) + y(t) = 1$ (Hint - you do not need to solve this differential equation).

   (c) The set of 2×2 symmetric matrices.

   (d) The set of invertible matrices.

   (e) The set of points (x, x^2) in $\mathbb{R}^2$.

6. If $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$ and $A = (v_1 \ldots v_n)$ is an $n \times n$ matrix with the i^{th} column as v_i, then what is the relation between $\det A$ and span $\{v_1, \ldots, v_n\}$?

7. Show that the set of solutions to $y''(t) = y'(t)$ is a vector space (Hint - you do not need to solve this differential equation).

8. Recall that the set S of polynomials of degree no more than 3 forms a vector space of dimension 4.

   (a) Show that

   $$b_1(t) = 1 + t, \qquad b_2(t) = 1 - t, \qquad b_3(t) = t^2, \qquad b_4(t) = 1 + t + t^2 + t^3$$

   forms a basis for S.

   (b) Express $1 - 2t^3$ in the given basis.

9. Let A be an $n \times n$ matrix, and let P be an $n \times n$ permutation matrix. How do the matrices AP and PA relate to A?

10. Let V be a vector space and $W \subset V$ be a subspace. Recall that a subspace $Z \subset V$ is a *complementary subspace* to W if

    (i) $W \cap Z = \{\mathbf{0}\}$, and

    (ii) $W + Z = V$, i.e., every $v \in V$ can be expressed as $v = w + z$, for $w \in W$ and $z \in Z$.

    Show that the y-axis and the line $y = x$ form complementary subspaces of $\mathbb{R}^2$.

11. Recall that a *projection matrix* is a square matrix P such that $P^2 = P$. Consider the matrix $A = \frac{1}{5}\begin{bmatrix} -1 & -2 \\ 3 & 6 \end{bmatrix}$.

    (a) Show that A is a projection matrix.

    (b) Find the image and kernel of A.

    (c) Prove that the image and kernel of A form complementary subspaces of $\mathbb{R}^2$.

12. Determine whether the following sets vectors (or functions) consist of linearly independent vectors. No explanation is necessary.

    (a) $\{e^x, e^{x+1}\}$

    (b) $\{e^x, xe^x, x^2e^x\}$

(c) $\{(1,2,1),(1,-1,2),(-1,-2,0),(1,2,3)\}$

(d) $\{\cos x, \sin x, \cos 2x, \sin 2x\}$

13. Let V be the space of polynomials over $\mathbb{R}$ in one variable of degree no more than 4. Let D be the differentiation map. Find the set S of all polynomials $p(x) \in V$ such that $D^2(p(x)) = 0$ (where the right hand side denotes the 0 function). Show that S is a subspace of V. Is the image of D a subspace of V?

14. Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix. Let O be the $n \times m$ matrix of all 0's. Consider the block matrix

$$K = \begin{bmatrix} A & O \\ O^T & B \end{bmatrix}$$

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(a) What are the dimensions of K?

(b) What is the rank of K in terms of the ranks of A, B, and O?

(c) What is the kernel of K in terms of the kernels of A, B, and O?

(d) What is $\det K$?

15. Recall that the *cokernel* of a matrix A is $\text{coker } A = \ker A^T$, and the *corange* of A is $\text{corng } A = \text{rng } A^T$. If $A : \mathbb{R}^n \to \mathbb{R}^m$ has rank r, what are the dimensions of the following spaces in terms of m, n, and r?

(a) $\ker A$

(b) $\text{rng } A$

(c) $\text{coker } A$

(d) $\text{corng } A$

16. Let V be a vector space and $S = \{v_1, \dots, v_n\}$ be a spanning set for V.

(a) Show that S contains a basis for V.

(b) If W is any subspace of V, must W have a basis in the set V? Prove or give a counter-example.