Review Problems for Midterm 2

July 6, 2015

Here is a list of review problems for midterm 2. This list is designed to help students understand the type of question which could be asked. This list is not exhaustive. Students are responsible for all of the material in the homework problems (especially the starred ones!), as well as material from lecture and material in the text from the sections covered.

1. Consider the following pairings $< v, w >$. Write them in matrix notation, that is, find matrices A, such that $< v, w >= v^T A w$. Which pairings define an inner product? (you should give an explanation to your answer).

   (a) $< v, w >= v_1 w_1 - v_1 w_2 - 2v_2 w_1 + 4v_2 w_2$

   (b) $< v, w >= 4v_1 w_1 + 2v_1 w_2 + 2v_2 w_1 + 4v_2 w_2 + v_3 w_3$

   (c) $< v, w >= (v_1 + v_2)(w_1 + w_2)$

2. Does the formula
$$< f, g >= \int_{-1}^{1} f(x) g(x) x \, \mathrm{dx}$$
define an inner product on the space $C^0[-1, 1]$ (of functions that are continuous on the interval $[-1, 1]$)?

3. Find a nonzero quadratic polynomial that is orthogonal to both $p_1(x) = 1$ and $p_2(x) = x$ under the L^2 inner product on the interval $[-1, 1]$.

4. Consider the pairing
$$< v, w >= 3v_1 w_1 + 2v_2 w_2 + v_3 w_3$$

   (a) Show that it defines an inner product.

   (b) Prove that
$$(3x_1 y_1 + 2x_2 y_2 + x_3 y_3)^2 \leq (3x_1^2 + 2x_2^2 + x_3^2) \cdot (3y_1^2 + 2y_2^2 + y_3^2)$$
   for all real numbers $x_1, x_2, x_3, y_1, y_2, y_3$.

   (c) Prove that
$$(3x_1^2 + 2x_2^2 + x_3^2) \leq 6 \cdot (3x_1^2 + 2x_2^2 + x_3^2)$$
   for all real numbers x_1, x_2, x_3.

5. Consider the pairing

$$< v, w >= v_1w_1 - v_1w_2 - v_2w_1 + 4v_2w_2$$

   (a) Write the pairing in matrix notation, that is, find a matrix A such that $< v, w >= v^T Aw$.

   (b) Show that the pairing $< v, w >$ defines an inner product.

   (c) Give the Cauchy-Schwartz inequality for the inner product $< v, w >$ defined above in terms of the coordinate v_i, w_i.

   (d) Prove that

$$(7w_2 - w_1)^2 \leq 13 \cdot (w_1^2 - 2w_1w_2 + 4w_2^2)$$

   for all real numbers w_1, w_2.

6. Consider the quadratic form

$$q(\boldsymbol{x}) = x^2 + 3xy + 3y^2 - 2xz + 8z^2$$

   (a) Express $q(\boldsymbol{x})$ in matrix notation, that is, find a matrix A, such that $q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$

   (b) Show that $q(\boldsymbol{x})$ is positive definite.

   (c) Write $q(\boldsymbol{x})$ as a sum of squares.

7. Find the closest point to $\boldsymbol{b} = (1, 1, 2, -2)^T$ in the subspace spanned by the vectors

$$\left\{(1, 2, -1, 0)^T, (0, 1, -2, -1)^T, (1, 0, 3, 2)^T\right\}$$

8. Find the straight line $y = \alpha + \beta t$ that best fits the following data in the least squares sense:

t_i	1	2	3	4	5
y_i	0	3	2	1	4

9. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$$

   (a) Use the definition of an orthogonal matrix to determine if A is orthogonal.

   (b) Compute the QR factorization of A.

10. Prove the following statements/identities:

   (a) If $< u, v >$ is an inner product in V, then

$$< u, v >= \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right)$$

   for any vectors u, v in V.

   (b) Suppose A and B are two $n \times n$ matrices and e_i is the i^{th} element of the standard basis of $V = \mathbb{R}^n$.

i. It is true that

$$e_i^T A e_j = A_{ij}$$

ii. If $v^T A w = v^T B w$ for all vectors v amd w in V, then $A = B$.

(c) The Cauchy-Schwarz inequality implies the triangle inequality. (That is, we are asking you to *prove* the triangle inequality, *using* the Cauchy-Schwarz inequality.)

(d) For any $A \in M_{n\times m}$, the matrix $A^T A$ is positive semi-definite.

(e) If for a matrix A we know that $A^T A$ is positive definite, then $\ker A = \{0\}$.

(f) For any matrix A, the matrix

$$P := A \cdot (A^T A)^{-1} \cdot A^T$$

is a projection matrix, that is, it satisfies $P^2 = P$.

(g) Let v and w be elements of an inner product space. Then,

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

if and only if v and w are orthogonal.

(h) Consider two matrices $A \in M_{k\times s},\ B \in M_{s\times t}$.

i. If $\operatorname{rank}(AB) = n$, then $\operatorname{rank} A \geq n$ and $\operatorname{rank}(B) \geq n$.

ii. If $< v, w >= v^T K w$ is an inner product for a space V and the vectors $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_n$ are orthogonal to each other, with respect to $< v, w >$, then they are linearly independent.

(i) Any unit vector $\boldsymbol{u}$ in some inner product space V is part of an orthonormal basis of V.

11. Consider the following vectors:

$$\boldsymbol{v}_1 = (1, 1, 1)^T \quad \boldsymbol{v}_2 = (1, 1, -2)^T \quad \boldsymbol{v}_3 = (-1, 1, 0)^T$$

(a) Prove that they are orthogonal to eachother.

(b) Use orthogonality to write the vector $\boldsymbol{v} = (1, 2, 3)^T$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

12. Find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

13. Let V be the plane in $\mathbb{R}^3$ given by $3x - y + z = 0$.

(a) Find an orthonormal basis for V.

(b) Find the coordinates of the point $\boldsymbol{p} = (2, 4, -2)$, of V, in terms of your basis.

14. Starting with the polynomials $1, x, x^2$, use the Gram-Schmidt process to determine an orthonormal basis for $P^{(2)}$ equipped with the usual L^2-inner product:

$$< p, q >= \int_0^1 p(x)q(x)\,\mathrm{dx}$$

15. Consider the subspace W of $\mathbb{R}^4$ given by

$$W := \ker \begin{pmatrix} 1 & 0 & 0 & 2 \\ -2 & -1 & 1 & -3 \end{pmatrix}$$

(a) Find a basis for the orthogonal complement $W^\perp$ of W.

(b) Decompose the vector $\boldsymbol{v} = (1,0,0,1)^T$ as $\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{z}$, where $\boldsymbol{w} \in W$ and $\boldsymbol{z} \in W^\perp$.

(c) What is the closest point to $\boldsymbol{v}$ in W?

16. Indicate which of the following statements are True and which are False. Write a short explanation for or give a counter-example to the starred statements.

* (a) i. If A and B are symmetric matrices, so is AB.

ii. If A and B are positive definite matrices, so is AB.

(b) Every diagonal matrix is positive-definite.

* (c) A gramm matrix is always symmetric.

(d) The Gramm matrix for a collection of orthogonal vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ is always diagonal.

(e) Any basis of $\mathbb{R}^n$ is an orthonormal basis with respect to some inner product.

* (f) The transpose of an orthogonal matrix is also orthogonal.

(g) An upper triangular matrix U is orthogonal if and only if U is a diagonal matrix.

* (h) If $W, Z \subset \mathbb{R}^n$ are complementary subspaces, then so are $W^\perp$ and $Z^\perp$.