Review Problems for Midterm 3

Here is a list of review problems for midterm 1. This list is designed to help students understand the type of question which could be asked. This list is not exhaustive. Students are responsible for all of the material in the homework problems (especially the starred ones!), as well as all material from lecture and material in the text from the sections covered.

1. Find a matrix representation for the following linear transformations on $\mathbb{R}^3$:
   (a) Counterclockwise rotation by $30°$ around the y-axis.
   (b) Orthogonal projection onto the plane $2x - y - z = 0$.
   (c) Reflection in the plane $x + 2y + 3z = 0$.
   (d) Reflection swapping the point $(1, 2, 2)^T$ with $(3, 0, 0)^T$.

2. Follow these steps to find the matrix representation for the clockwise rotation by $60°$ around the line $W = \{(x, y, z)^T \ : \ x = 2y = z\}$ in the standard basis of $\mathbb{R}^3$. Let L be this linear transformation.
   (a) Find orthonormal bases β_1 for W and β_2 for $W^\perp$.
   (b) Let $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\} = \beta_1 \cup \beta_2$ be the orthonormal basis for $\mathbb{R}^3$ obtained by combining the orthonormal bases for W and $W^\perp$ (where w_1 spans W and $\boldsymbol{w_2}$ and $\boldsymbol{w_3}$ span $W^\perp$). Calculate the determinant of the orthogonal matrices $S = (\boldsymbol{w_1 w_2 w_3})$ and $R = (\boldsymbol{w_1 w_3 w_2})$, and indicate which of S and R is proper orthogonal and which is not. The ordering in which the matrix is proper orthogonal makes the corresponding coordinate system right handed.
   (c) Write the matrix representation for L in the $\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}$ basis, using the right-handed order of vectors you found in part (b).
   (d) Let X be the proper orthogonal matrix from part (b), and let A be the matrix from part (c). Calculate the matrix B for L in the standard basis as $B = XAX^{-1}$.
   (e) If C is the matrix from (b) which is not proper orthogonal, what transformation does XBX^{-1} represent?

3. For each of the following linear maps, describe all of their real eigenvalues and the corresponding eigenvectors geometrically. You do not need to write down any matrices, although you should give specific eigenvalues.
   (a) A $30°$ rotation about the x-axis in $\mathbb{R}^3$.
   (b) A projection onto the plane $x + y + z = 0$ in $\mathbb{R}^3$.
   (c) The horizontal shear map in $\mathbb{R}^2$ sending $\boldsymbol{e_1}$ to $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ to $\boldsymbol{e_1} + \boldsymbol{e_2}$.
   (d) The reflection in $\mathbb{R}^3$ in the plane $x + 2y - z = 0$.

4. Let P denote the orthogonal projection of $\mathbb{R}^3$ onto the plane $V = \{(x, y, z)^T \ : \ x + y - z = 0\}$, and let Q denote the orthogonal projection of $\mathbb{R}^3$ onto the plane $W = \{(x, y, z)^T \ : \ x - y - z = 0\}$. Finally, let R denote the orthogonal projection onto the line $V \cap W$.
   (a) Write the matrix representations of P, Q, and R.
   (b) Does $Q \circ P = R$? Does $P \circ Q = R$?

(c) Do P and Q commute?

5. Write the dual basis for the vectors $(1,2)^T, (2,1)^T$ in $\mathbb{R}^2$.

6. Indicate which of the following statements are True and which are False. Write a short explanation for or give a counter-example to the starred statements.

   *(a) If V is an n dimensional vector space, then V^* has dimension n.

   (b) $\mathcal{L}(\mathcal{P}^{(1)}, \mathcal{P}^{(3)})$ has dimension 2.

   (c) The matrix representation for a 45° rotation around the line $x = 2y = -17z$ in $\mathbb{R}^3$ (with the standard basis) is orthogonal.

   (d) The orthogonal projection in $\mathbb{R}^3$ onto the plane $-x + y + z = 0$ can be represented by an orthogonal matrix.

   (e) Every invertible linear transformation on $\mathbb{R}^2$ sends the unit square to a parallelogram.

   (f) If A and B are similar matrices and A is orthogonal, so is B.

   (g) If A and B are similar matrices and A is symmetric, so is B.

   *(h) Let A and B be $n \times n$ matrices, and let S be an orthogonal $n \times n$ matrix. If A is symmetric and $B = S^{-1}AS$, then B is symmetric.

   * (i) If A and B are similar matrices, then $\det A = \det B$.

   (j) If v is an eigenvector for a matrix A with eigenvalue 2, then $3v$ is an eigenvector for A with eigenvalue 6.

   (k) If A is an $n \times n$ matrix with real entries, A has all real eigenvalues.

   (l) If v_1 and v_2 are λ-eigenvectors for a matrix A, then so is $3v_1 + 2v_2$.

7. Prove the following statements.

   (a) If u is a vector in $\mathbb{R}^n$ with norm 1, then $(I - 2uu^T)^2 = I$.

   (b) If S is a 3×3 orthogonal matrix and R is the 3×3 matrix representing a rotation about the y axis, then $S^{-1}RS$ represents a rotation matrix. What is the axis of rotation?

   (c) If λ is an eigenvalue for a matrix A, λ^2 is an eigenvalue for A^2.

   (d) If A is a matrix with real entries and $z = a + bi$ is a complex eigenvalue for A, so is $\overline{z} = a - bi$.

   (e) Let A and B are $n \times n$ matrices such that $AB = BA$. Show that if $Av = 5v$ and all other $5-$eigenvectos for A are multiples of v, v is also an eigenvector for B.

   (f) If A is a 2×2 matrix, then $A^2 - (\text{tr } A)A + (\det A)I = 0$.

8. Diagonalize the following matrices.

   (a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

   (b) $B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ -1 & 4 & 2 \end{bmatrix}$.

9. Consider the following orthonormal basis of $\mathbb{R}^3$:

$$u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad u_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad u_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

Let P be the plane orthogonal to u_1, and let L be the reflection in the plane P.

(a) Write the matrix for L in both the standard basis and the $\{u_1, u_2, u_3\}$ basis.

(b) What are the eigenvalues and eigenvectors of L?

10. Consider the matrix

$$A = \begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix}.$$

(a) Find all eigenvalues and eigenvectors of A.

(b) Is A diagonalizable? If not, why not?

11. Find the characteristic polynomials of the following matrices. Indicate where the trace and determinant of these matrices appears in the corresponding polynomials.

(a) $A = \begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$

(b) $B = \begin{bmatrix}1 & -1 & 2\\0 & 2 & 1\\1 & 1 & 1\end{bmatrix}$

(c) The block diagonal matrix $D = \begin{bmatrix}A & O\\O & B\end{bmatrix}$, where O denotes a matrix of all 0's with the appropriate dimensions.

12. Let $V \subset C^1(\mathbb{R})$ be the vector space spanned by the functions $\cos\theta$ and $\sin\theta$, and let D be the differentiation map. Write the matrix for D in the basis $\{\cos\theta, \sin\theta\}$.

13. Let $\beta_1 = \{\boldsymbol{v_1} = (1, 1, -2)^T, \boldsymbol{v_2} = (-1, 1, 0)^T, \boldsymbol{v_3} = (2, 0, 1)^T\}$ and $\beta_2 = \{\boldsymbol{w_1} = (1, 0, 1)^T, \boldsymbol{w_2} = (1, 2, 1)^T, \boldsymbol{w_3} = (1, 0, -1)^T\}$ be two bases for $\mathbb{R}^3$.

(a) Construct a matrices S_1 and S_2 such that S_1 sends the standard basis to the β_1 basis, and S_2 sends the standard basis to the β_2 basis. That is, $S_1\boldsymbol{e_1} = \boldsymbol{v_1}, S_1\boldsymbol{e_2} = \boldsymbol{v_2}, S_1\boldsymbol{e_3} = \boldsymbol{v_3}$, and $S_2\boldsymbol{e_1} = \boldsymbol{w_1}, S_2\boldsymbol{e_2} = \boldsymbol{w_2}, S_2\boldsymbol{e_3} = \boldsymbol{w_3}$.

(b) Construct a matrix S sending the β_1 basis to the β_2 basis.

(c) The matrix $M = \frac{1}{3}\begin{bmatrix}2 & 1 & -2\\1 & -4 & 8\\1 & -1 & 5\end{bmatrix}$ represents a linear transformation L in the β_1 basis. What is the matrix for L in the β_2 basis?