Math 4242
Summer 2020
PRACTICE EXAM
07/10/2020
Time Limit: Up to you

Name (Print): ____________________

Instructor ____________________

Exam 2 contains 3 pages (including this cover page) and 10 problems. Please check to see if any pages are missing.

There are 10 problems instead of 5, but break up this exam into 2 parts 1-5 and 6-10, if you want to practice exams more authentically.

Show your work on each problem. Specifically:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Unsupported answers will not receive credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will likely receive partial credit.
- **Circle your final answer** for problems involving a series of computations.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Please do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. (20 points) Let A be any square matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.
   (a) (5 points) Show that the quadratic forms $x^T A x$ and $x^T A^T x$ are equal.
   (b) (5 points) Show that $K = \frac{1}{2}(A + A^T)$ is a symmetric matrix.
   (c) (5 points) Conclude that it suffices to only consider quadratic forms of symmetric matrices by showing that $x^T A x = x^T K x$.
   (d) (5 points) Prove that if K is positive definite, then every diagonal entry of A is positive.

2. (20 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 4y \\ 2y \end{pmatrix}.$$
   Rewrite this transformation using coordinates in the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

3. (20 points) Let $f(x) = 1$ and $g(x) = ax$ for $a \neq 0$ in the vector space $C^0[0,1]$ with inner product
$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$
   (a) (10 points) Find the $a \in \mathbb{R}$ such that the angle between f and g is $\pi/6$, or 30 degrees.
   (b) (10 points) Does your answer change if $f(x) = b$ for some other $b \neq 0, 1$? Explain why or why not.

4. (20 points) Let v and w be independent vectors in $\mathbb{R}^n$. Let $v^\perp$ and $w^\perp$ denote the orthogonal subspaces of span(v) and span(w). Show that $\dim(v^\perp \cap w^\perp) = n - 2$.

5. (20 points) Find an orthonormal basis for the subspace
$$W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}\right\}.$$

6. (20 points) Let $P = I - uu^T$ where u is a unit vector in $\mathbb{R}^n$ with the dot product.
   (a) (10 points) Show by direct computation that $P^2 = P$.
   (b) (10 points) Compute $(\mathrm{img}(P))^{\perp}$.

7. (20 points) Find the distance between $v = (0, 3, -2, -2)$ and $w = (4, -1, 2, 1)$ in the following norms on $\mathbb{R}^4$.
   (a) (6 points) The L^2 norm
   (b) (7 points) The L^1 norm
   (c) (7 points) The L^∞ norm

8. (20 points) For the following statements, list whether they are true or false. If false, provide a counterexample.
   (a) (5 points) All matrices with positive entries are positive definite.
   (b) (5 points) Let $A = \begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix}$. Then $\|A\|_\infty = 5$.
   (c) (5 points) All norms satisfy the parallelogram identity.
   (d) (5 points) Let A and B be $n \times n$ matrices representing the same transformation $\mathbb{R}^n \to \mathbb{R}^n$ in two different bases. Then $\det A \neq \det B$.

9. (20 points) Let $T : V \to W$ be a linear transformation. Define the kernel of T to be

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

   (a) (10 points) Show that $\ker T$ is a subspace of V. (Notice this is a generalization of the matrix case.)
   (b) (10 points) Assume that $\ker T = 0$. Show that if $T(v) = T(w)$, then $v = w$.

10. (20 points) Prove that a matrix K is positive definite iff for all nonzero $v \in \mathbb{R}^n$ that the angle θ between v and Kv is acute, i.e. $|\theta| < \pi/2$.