**1.** Let $A$ be any square matrix in $\mathcal{M}_{n\times n}(\mathbb{R})$.

(a) (5 points) Show that the quadratic forms $x^TAx$ and $x^TA^Tx$ are equal.

(b) (5 points) Show that $K = \frac{1}{2}(A + A^T)$ is a symmetric matrix.

(c) (5 points) Conclude that it suffices to only consider quadratic forms of symmetric matrices by showing that $x^TAx = x^TKx$.

(d) (5 points) Prove that if $K$ is positive definite, then every diagonal entry of $A$ is positive.

*Solution.* (a) This follows from the fact that the dot product is symmetric.

$$x^TAx = x \cdot Ax = Ax \cdot x = (Ax)^Tx = x^TA^Tx$$

(b) This is symmetric by the fact that constants come out of transposes and that the transpose twice is the original matrix.

$$K^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = K$$

(c) By the first two parts

$$x^TKx = x^T\left(\frac{1}{2}(A + A^T)\right)x = \frac{1}{2}\left(x^TAx + x^TA^Tx\right) = \frac{1}{2}\left(2x^TAx\right) = x^TAx.$$

(d) If $K$ is positive definite, then $x^TKx > 0$ for all nonzero $x$. In particular, let $x = e_i$. On the one hand $e_i^TAe_i = a_{ii}$, the $i$th diagonal entry. But on then other hand $e_i^TAe_i = e_i^TKe_i > 0$. Therefore $a_{ii} > 0$.

**2.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 4y \\ 2y \end{pmatrix}.$$

Rewrite this transformation using coordinates in the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

*Solution.* First, we cna notice that this is a transformation in standard coordinates given by

$$A = \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix}.$$

By the change of basis formula, the transformation is represented by a matrix $B = S^{-1}AS$ in coordinate $v_1 = (1, 0)$ and $v_2 = (4, 3)$, where

$$S = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}.$$

Computing this term out

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

**3.** Let $f(x) = 1$ and $g(x) = ax$ for $a \neq 0$ in the vector space $C^0[0, 1]$ with inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) (10 points) Find the $a \in \mathbb{R}$ such that the angle between $f$ and $g$ is $\pi/6$, or 30 degrees.

(b) (10 points) Does your answer change if $f(x) = b$ for some other $b \neq 0, 1$? Explain why or why not.

*Solution.* (a) Recall that $\|1\| \, \|ax\| \cos(\theta) = \langle 1, ax \rangle$, so that

$$\cos(\theta) = \frac{\langle 1, ax \rangle}{\|1\| \, \|ax\|} = \frac{\int_0^1 ax \, dx}{\sqrt{\int_0^1 1 \, dx}\sqrt{\int_0^1 a^2x^2 \, dx}} = \frac{\frac{1}{2}a}{1\sqrt{\frac{a^2}{3}}} = \frac{\sqrt{3}}{2}\frac{a}{|a|}.$$

Note that if $a > 0$, then $a/|a| = 1$ and if $a < 0$ then $a/|a| = -1$. But $\cos \pi/6 = \sqrt{3}2$ so we want the positive solution. Therefore for all $a > 0$, then the angle between $f = 1$ and $g = ax$ is $\theta = \pi/6$.

(b) We can answer this question using the general principles of inner products. Indeed

$$\cos(\theta) = \frac{\langle b, ax \rangle}{\|b\| \, \|ax\|} = \frac{ab\langle 1, x \rangle}{|a||b| \, \|1\| \, \|x\|} = \frac{\sqrt{3}}{2}\frac{a}{|a|}\frac{b}{|b|}.$$

Therefore we see that $a$ and $b$ have the same sign, then the angle is 30 degrees, and if they have opposite sign, the angle is 150 degrees. This makes sense geometrically. The angle between $v$ and $w$ should be the same as with $v$ and $cw$ for $c > 0$. And if $c < 0$, then the angle becomes $\pi - \theta$. Scaling these vectors shouldn't change the angle unless the scale changes the sign. This principle is just being applied to functions in this problem.

**4.** Let $v$ and $w$ be independent vectors in $\mathbb{R}^n$. Let $v^\perp$ and $w^\perp$ denote the orthogonal subspaces of $\text{span}(v)$ and $\text{span}(w)$. Show that $\dim (v^\perp \cap w^\perp) = n - 2$.

*Solution.* Indeed $v^\perp \cap w^\perp = \text{span}(v, w)^\perp$, namely the subspace of vectors orthogonal to both $v$ and $w$. Putting $v$ and $w$ into the columns of an $n \times 2$ matrix A, we know that

$$v^\perp \cap w^\perp = \text{span}(v, w)^\perp = \ker A^T.$$

Since $v$ and $w$ are independent, the rank of $A^T$ is 2, and therefore the kernel has dimension $n - 2$ by rank nullity. This completes the proof.

**5.** Find an orthonormal basis for the subspace

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

*Solution.* This is the Gram-Schmidt process. We can do the normal version, and then divide the resulting orthogonal basis by the vectors' norms to get an orthonormal one. First

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ -2 \\ 2 \\ 0 \end{pmatrix}.$$

Actually by inspection, the third vector is already orthogonal to these two. Therefore $v_3 = (0, 0, 0, -1)$. Therefore an orthonormal basis is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

**6.** Let $P = I - uu^T$ where $u$ is a unit vector in $\mathbb{R}^n$ with the dot product.

(a) (10 points) Show by direct computation that $P^2 = P$.

(b) (10 points) Compute $(\mathrm{img}(P))^\perp$.

*Solution.* (a) Note that since $u$ is a unit vector, then $u^T u = 1$. Therefore

$$\begin{aligned} P^2 = (I - uu^T)(I - uu^T) = I - 2uu^T + (uu^T)(uu^T) &= I - 2uu^T + u(u^T u)u^T \\ &= I - 2uu^T + uu^T = I - uu^T = P. \end{aligned}$$

(b) By the relationship between the fundamental subspace, $(\mathrm{img}(P))^\perp = \mathrm{coker} P = \ker P^T$. First,

$$P^T = (I - uu^T)^T = I^T - (uu^T)^T = I - (u^T)^T u^T = I - uu^T = P.$$

Turns out $P$ is symmetric. Therefore, we need to find $\ker P$. In fact $\mathrm{span}(u) = \ker P$. If we let $w \in \ker P$, then $(I - uu^T)w = 0$. But expanding this out gives that equation $w - uu^T w = 0$, which tells us that

$$W = u(u^T w) = au$$

where $a = u \cdot w$. Therefore $w$ must be a multiple of $u$. Furthermore $u \in \ker P$, since

$$(I - uu^T)u = u - u(u^T u) = u - u = 0.$$

Therefore $\mathrm{img}(P)^\perp = \ker P$ is the span of $u$.

**7.** Find the distance between $v = (0, 3, -2, -2)$ and $w = (4, -1, 2, 1)$ in the following norms on $\mathbb{R}^4$.

(a) (6 points) The $L^2$ norm

(b) (7 points) The $L^1$ norm

(c) (7 points) The $L^\infty$ norm

*Solution.* This distance between two vectors is always $\|v - w\|$. To make this easier, $v - w = (-4, 4, -4, -3)$.

(a) $\|(-4, 4, -4, -3)\|_2 = \sqrt{16 + 16 + 16 + 9} = \sqrt{57}$
(b) $\|(-4, 4, -4, -3)\|_1 = |-4| + |4| + |-4| + |-3| = 15$
(c) $\|(-4, 4, -4, -3)\|_\infty = \max\{|-4|, |4|, |-4|, |3|\} = 4$

**8.** For the following statements, list whether they are true or false. If false, provide a counterexample.

(a) (5 points) All matrices with positive entries are positive definite.

(b) (5 points) Let $A = \begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix}$. Then $\|A\|_\infty = 5$.

(c) (5 points) All norms satisfy the parallelogram identity.

(d) (5 points) Let $A$ and $B$ be $n \times n$ matrices representing the same transformation $\mathbb{R}^n \to \mathbb{R}^n$ in two different bases. Then $\det A \neq \det B$.

*Solution.* (a) False, $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not positive definite. The quadratic form is $x^2 + 4xy + y^2$, which is not positive when $(x, y) = (1, -1)$, since $q(1, -1) = 1^2 + -4 + 1^2 = -2$.
(b) True, the infinity norm on matrices is the max row sum, which in this case is 5. (You can just put true, thought I'd just explain here.)
(c) False, the $L^1$ norms and $L^\infty$ norms do not satisfy the parallelogram identity. Take $V = \mathbb{R}^2$ with the $L^1$ norm. If $v = (1, 0)$ and $w = (0, 1)$ then

$$\|v + w\|_1^2 + \|v - w\|_1^2 = 8$$

but

$$2\|v\|_1^2 + 2\|w\|_1^2 = 4.$$

(d) False, they in fact are always equal determinants.

$$\det B = \det S^{-1}AS = \det S^{-1} \det A \det S = \frac{1}{\det S} \det A \det S = \det A.$$

We've done this exercise, but now you know what it's for! The determinant of a transformation doesn't depend on what coordinates you use.

**9.** Let $T : V \to W$ be a linear transformation. Define the kernel of $T$ to be

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

(a) (10 points) Show that $\ker T$ is a subspace of $V$. (Notice this is a generalization of the matrix case.)

(b) (10 points) Assume that $\ker T = 0$. Show that if $T(v) = T(w)$, then $v = w$.

*Solution.* (a) We show that $\ker T$ is nonempty, closed under sums, and closed under scalar multiplication. First $0 \in \ker T$, since $T(0) = 0$ always. Second, if $T(v) = 0$ and $T(w) = 0$, then

$$T(v + w) = T(v) + T(w) = 0 + 0 = 0$$

so that $v + w \in \ker T$. Finally

$$T(cv) = cT(v) = c \cdot 0 = 0$$

so that $cv \in \ker T$. Therefore $\ker T$ is a subspace.

(b) Note that $T(v) - T(w) = T(v - w)$ so that if $T(v) = T(w)$, then $T(v - w) = 0$. So $v - w \in \ker T$. But $\ker T = 0$, so that $v - w = 0$. Therefore $v = w$ as desired.

**10.** Prove that a matrix $K$ is positive definite iff for all nonzero $v \in \mathbb{R}^n$ that the angle $\theta$ between $v$ and $Kv$ is acute, i.e. $|\theta| < \pi/2$.

*Solution.* Remember that an angle $\theta$ is acute iff $\cos(\theta) > 0$, since cos is positive only when $\pi/2 < \theta < \pi/2$. Furthermore if $\theta$ is the angle between $v$ and $Kv$, then $\theta$ is acute iff

$$\cos(\theta) = \frac{\langle v, Kv \rangle}{\|v\| \, \|Kv\|} = \frac{v \cdot Kv}{\|v\| \, \|Kv\|} = \frac{v^T K v}{\|v\| \, \|Kv\|} > 0$$

So we need to show that $K$ is positive definite iff

$$\frac{v^T K v}{\|v\| \, \|Kv\|} > 0$$

for all nonzero $v \in \mathbb{R}^n$.

This is a bit more straightforward. If $K$ is positive definite, then $v^T K v > 0$. Furthermore $Kv \neq 0$, since being positive definite implies that $K$ is invertible, which implies that $\ker K = 0$. Therefore this fraction is well defined (no dividing by 0), since $\|v\| > 0$ and $\|Kv\| > 0$. Since the numerator and denominator are both positive, then entire fraction is strictly positive as well.

Conversely, If the fraction is strictly positive, then it is well-defined (no dividing by 0). We know the denominator is then positive since it is the norms of some vectors, and norms are always positive. Finally since the whole fraction is positive, and so is the denominator, then the numerator must be positive for all $v \neq 0$. Therefore $v^T K v > 0$ for all $v \neq 0$ and by definition $K$ is positive definite.