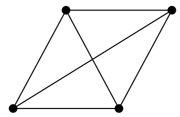
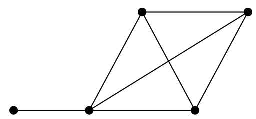
1. Consider the following graph.



Use the Euler characteristic formula to calculate the dimension of the cokernel of the incidence matrix. Can you identify the independent circuits on the graph visually?

2. Consider the graph from problem 1 but with one edge attached to it. (It looks like a kite!)



Suppose we are studying Markov process associated to a random walk on this graph.

(a) Write down the transition matrix for this problem. The transition matrix is regular, but which power of the transition matrix has all nonzero entries?

(b) What is the probability that a random walk on this graph will be at any given vertex?

(Hint: It may be annoying to work with a 5×5 matrix without a computer, but use your knowledge of the situation to work around it! For example, you already know that $\lambda = 1$ is an eigenvalue since it is regular. So all you have to do is find the eigenvectors V_1. Don't worry I won't ask you to row reduce a 5×5 on the exam. $n = 3$ at most. This is just practice.)

3. Consider the system $Ax = b$ where

$$A = \begin{pmatrix} 0 & 1 \\ -3 & 1 \\ 2 & 2 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

(a) Find the least squares solution to this system.

(b) Which element $w^* \in \operatorname{img}(A)$ actually is at minimum distance from b?

4. Consider the matrix

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix}.$$

(a) Without even doing any calculation, you should be able to look at this matrix and know one of the eigenvalues. What is that eigenvalue and why?

(b) Calculate $\|A\|_\infty$. Does it imply that $A^k \to 0$?

(c) Show properly that $A^k \to 0$ as $k \to \infty$.

5. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{pmatrix}.$$

6. Find the Schur decomposition of the matrix

$$B = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

What about its Jordan decomposition?

7. Find the Jordan decomposition of the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 9 & -4 & -3 \\ 0 & 0 & -1 \end{pmatrix}.$$

8. Compute the spectral decomposition of the matrix

$$D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

What about the QR decomposition?

9. Let B be a positive definite symmetric matrix. Suppose B^2 has spectral decomposition

$$B^2 = Q\Lambda Q^T.$$

Find a spectral decomposition of B in terms of Q and Λ.

10. Suppose A is a square matrix with two different diagonalizations

$$S\Lambda S^{-1} = A = T\Lambda' T^{-1}.$$

Do Λ and Λ' have to be equal matrices? If not, what do they have in common? What about S and T?

11. Consider the linear iterative system $u^{(0)} = (1, 0, 1)$, and $u^{(k+1)} = Tu^{(k)}$ where

$$T = \frac{1}{6}\begin{pmatrix} 4 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & -9 & 3 \end{pmatrix}.$$

(a) Find all the fixed points of T.

(b) Compute the limit of $u^{(k)}$ as $k \to \infty$.