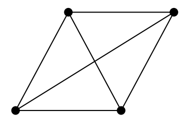
1. Consider the following graph.



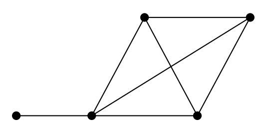
Use the Euler characteristic formula to calculate the dimension of the cokernel of the incidence matrix. Can you identify the independent circuits on the graph visually?

Solution. First, the dimension of the cokernel is the number of independent circuits in the We know that by the Euler characteristic formula

$$\dim \operatorname{coker}(A) = 1 - \text{number of vertices} + \text{number of edges} = 1 - 4 + 6 = 3.$$

Visually, the three independent circuits are the equalateral triangle in the bottom left, and the two longer triangles down the middle.

2. Consider the graph from problem 1 but with one edge attached to it. (It looks like a kite!)



Suppose we are studying Markov process associated to a random walk on this graph.

(a) Write down the transition matrix for this problem. The transition matrix is regular, but which power of the transition matrix has all nonzero entries?

(b) What is the probability that a random walk on this graph will be at any given vertex?

(Hint: It may be annoying to work with a 5×5 matrix without a computer, but use your knowledge of the situation to work around it! For example, you already know that $\lambda = 1$ is an eigenvalue since it is regular. So all you have to do is find the eigenvectors V_1. Don't worry I won't ask you to row reduce a 5×5 on the exam. $n = 3$ at most. This is just practice.)

Solution. (a) Labeling the vertices as 1-5 from top left to bottom right, the translation matrix is

$$T = \begin{pmatrix} 0 & 1/3 & 0 & 1/4 & 1/3 \\ 1/3 & 0 & 0 & 1/4 & 1/3 \\ 0 & 0 & 0 & 1/4 & 0 \\ 1/3 & 1/3 & 1 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/4 & 0 \end{pmatrix}.$$

Since you can get to every other vertex from any starting vertex after 3 steps, then T^3 has no nonzero entries. (T^2 doesn't work because you can't get from vertex 3 to vertex 4 in two steps.)

(b) We know that this matrix is regular so it has $\lambda = 1$ as a nonrepeating eigenvalue, so we can find the probability eigenvector in $\ker T - I$. This kernel is generated by $v = (3, 3, 1, 4, 3)$, which we can calculate by row reduction, so the corresponding probability vector is

$$u^* = \frac{1}{14}\begin{pmatrix} 3 \\ 3 \\ 1 \\ 4 \\ 3 \end{pmatrix}.$$

This represents the probability of the random walk being in each vertex.

3. Consider the system $Ax = b$ where

$$A = \begin{pmatrix} 0 & 1 \\ -3 & 1 \\ 2 & 2 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

(a) Find the least squares solution to this system.

(b) Which element $w^* \in \operatorname{img}(A)$ actually is at minimum distance from b?

Solution. (a) The least squares solution is

$$x^* = (A^T A)^{-1} A^T b = \frac{1}{77}\begin{pmatrix} 6 & -1 \\ -1 & 13 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \frac{1}{77}\begin{pmatrix} 1 \\ -13 \end{pmatrix}.$$

(b) The actual closest point from the image of A to B is

$$w^* = Ax^* = \frac{1}{77}\begin{pmatrix} -13 \\ -16 \\ -24 \end{pmatrix}.$$

4. Consider the matrix

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 0 \\ -1 & 2 & 0 \end{pmatrix}.$$

(a) Without even doing any calculation, you should be able to look at this matrix and know one of the eigenvalues. What is that eigenvalue and why?

(b) Calculate $\|A\|_\infty$. Does it imply that $A^k \to 0$?

(c) Show properly that $A^k \to 0$ as $k \to \infty$.

Solution. (a) This matrix has two rows which are exactly the same. Therefore the rows are dependent. Therefore the rank of A^T is less than 3, which means the rank of A is less than 3. The main theorem for invertible matrices tells us that therefore the $\ker A \neq 0$. Thus $\lambda = 0$ is eigenvalue.

(b) The L^∞ norm of A is the greatest absolute row sum. Adding the absolute values of the row entries gives that $\|A\|_\infty = 1$. This does not imply that $A^k \to 0$ since $\|A\|_\infty \geq 1$ and not < 1.

(c) To show that $A^k \to 0$, we need to show that all the eigenvalues have $|\lambda_i| < 1$. The characteristic polynomial is

$$-\lambda^3 + \lambda^2 - \frac{2}{9}\lambda = 0$$

so that the eigenvalues are $\lambda = 0, 1/3, 2/3$. All of these have absolute value less than 1, so the theorem tells us that $A^k \to 0$.

5. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{pmatrix}.$$

Solution. To diagonalize a matrix, we just need the eigenvalues and a basis of eigenvectors. The eigenvalues are $\lambda = -1, 1, 1$ with eigenvectors $v_1 = (2, -3, 1)$, $v_2 = (0, -1, 1)$, and $v_3 = (1, 0, 0)$. Putting these into the columns of a matrix, we get the change of basis transformation

$$S = \begin{pmatrix} 2 & 0 & 1 \\ -3 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So the diagonalization is $A = S\Lambda S^{-1}$.

6. Find the Schur decomposition of the matrix

$$B = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

What about its Jordan decomposition?

Solution. To find the Schur decomposition, we pick an eigenvector of B, and then make it into an orthonormal basis. We see that the eigenvalues of B are $\lambda = 0, -4$. Let's pick $\lambda = 0$, which has eigenvector $v = (1, 2)$. The corresponding unit vector is $u = \frac{1}{\sqrt{5}}(1, 2)$. We can complete this into an orthonormal basis by $u_2 = \frac{1}{\sqrt{5}}(-2, 1)$. Then putting u, u_2, into the columns of a matrix U, we can diagonalize the first column of B, we get that

$$U^T BU = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 0 & -4 \end{pmatrix} = \Delta.$$

Now we are done since Δ is an upper triangular matrix with the eigenvalues on the diagonal and U is orthogonal (or unitary but with real numbers), so

$$B = U\Delta U^T.$$

For the Jordan decomposition, this matrix is already diagonalizable, so the Jordan form is just the diagonal matrix. The two eigenvectors are $v = (1, 2)$ and $v = (-1, 2)$. Therefore the Jordan decomposition (aka diagonalization in this case) is

$$B = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}^{-1}.$$

7. Find the Jordan decomposition of the matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 9 & -4 & -3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution. Finding the eigenvalues, we get that we have a triple eigenvalue $\lambda = -1$. But the only eigenvector is $v = (1, 3, 0)$. So we need to generalized eigenvectors to complete the Jordan chain v, w_1, w_2. First, we can find w_1 by solving

$$(C - (-1)I)w_1 = v.$$

Solving this system gives a solution

$$w_1 = \frac{y}{3} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \\ 0 \end{pmatrix}.$$

We don't want vectors already in the kernel, so we pick $w_1 = (1/3, 0, 0)$. Now w_2 completes the chain, so we solve $(C - (-1)I)w_2 = w_1$, which gives us

$$w_2 = \frac{y}{3} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/9 \\ 0 \\ 1/3 \end{pmatrix}.$$

So we can pick $w_2 = (1/9, 0, 1/3)$. Therefore the Jordan decomposition is $C = SJS^{-1}$ where

$$S = \begin{pmatrix} 1 & 1/3 & 1/9 \\ 3 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

and

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

8. Compute the spectral decomposition of the matrix

$$D = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

What about the QR decomposition?

Solution. This is a symmetric matrix, so we can make an orthonormal basis of eigenvectors. This is the spectral decomposition. Find the eigenvectors normally, we get that $\lambda = -1, -2, 4$ with eigenvectors $v_1 = (0, 1, 0), v_2 = (1, 0, -1), v_3 = (1, 0, 1)$. Making these unit vectors we get that the orthogonal change of basis matrix is

$$Q = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The spectral decomposition is $D = Q\Lambda Q^T$.

The QR decomposition is different. To calculate it, you do alternate G-S on the columns of D, so the resulting orthonormal basis forms Q and the coefficients from G-S make R. Remember that alternate G-S relies on the recursive process. Let $w_1 = (1, 0, 3)$, $w_2 = (0, -1, 0)$, and $w_3 = (3, 0, 1)$. The basis of unit vectors we will get at the end is u_1, u_2, u_3. First, $w_1 = r_{11}w_1$, so that $r_{11} = \|w_1\| = \sqrt{10}$ and $u_1 = \frac{1}{\sqrt{10}}(1, 0, 3)$.

Then $w_2 = r_{12}u_1 + r_{22}u_2$. We know that $r_{12} = w_2 \cdot u_1 = 0$ and therefore $u_2 = w_2$ since w_2 is already a unit vector orthogonal to u_1. We saw that $r_{12} = 0$ and $r_{22} = 1$.

Now onto $w_3 = r_{13}u_1 + r_{23}u_2 + r_{33}u_3$. First $r_{13} = w_3 \cdot u_1 = \frac{6}{\sqrt{10}}$, and $r_{23} = w_3 \cdot u_2 = 0$. Finally

$$r_{33} = \sqrt{\|w_3\|^2 - r_{13}^2 - r_{23}^2} = \sqrt{10 - 36/10} = \sqrt{64/10} = \frac{8}{\sqrt{10}}.$$

Finally

$$u_3 = \frac{w_3 - r_{13}u_1 - r_{23}u_2}{r_{33}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}.$$

Therefore the QR decomposition is

$$D = \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \\ 0 & -1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 & \frac{6}{\sqrt{10}} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{8}{\sqrt{10}} \end{pmatrix}.$$

9. Let B be a positive definite symmetric matrix. Suppose B^2 has spectral decomposition

$$B^2 = Q\Lambda Q^T.$$

Find a spectral decomposition of B in terms of Q and Λ.

Solution. Let Λ have diagonal entries λ_i, i.e. the eigenvalues of B^2. Let μ_i be the eigenvalues of B. Since B is positive definite then it has all positive eigenvalues, so $\mu_i > 0$. But the eigenvalues of B^2 are the eigenvalues of B, but squared, so $\mu_i^2 = \lambda_i$ with the same eigenvectors. So the change of basis matrix Q is still the same, but the matrix of eigenvalues is the positive square root of what it used to be, $\mu_i = +\sqrt{\lambda_i}$. We can denote the diagonal matrix with the mu_i as $\sqrt{\Lambda}$. Therefore

$$B = Q\sqrt{\Lambda}Q^T.$$

10. Suppose A is a square matrix with two different diagonalizations

$$S\Lambda S^{-1} = A = T\Lambda' T^{-1}.$$

Do Λ and Λ' have to be equal matrices? If not, what do they have in common? What about S and T?

Solution. Neither the diagonal entries nor the change of basis matrices need be the same. While the eigenvalues have to be the same, we can rearrange the order of them. So Λ and Λ' are the same, but the eigenvalues can be in a different order. If we rearrange the eigenvalues, then we also need to rearrange the eigenvectors, so S and T could have their columns permuted. We can also change S by scaling any eigenvector, or in general changing the basis of eigenvectors for any eigenspace V_λ. Therefore S and T definitely can be different.

Try finding two different diagonalizations of the matrix from Problem 5.

11. Consider the linear iterative system $u^{(0)} = (1, 0, 1)$, and $u^{(k+1)} = Tu^{(k)}$ where

$$T = \frac{1}{6}\begin{pmatrix} 4 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & -9 & 3 \end{pmatrix}.$$

(a) Find all the fixed points of T.

(b) Compute the limit of $u^{(k)}$ as $k \to \infty$.

Solution. (a) All the fixed points are all the eigenvectors for the eigenvalues $\lambda = 1$, i.e. $\ker T - I$. Using row reduction, the kernel of $T - I$ is actually trivial, so $\lambda = 1$ is not an eigenvalue and the only fixed point is $u = 0$.

(b) But from part we computed the eigenvalues to be $\lambda = 1/2, 1/2 \pm i/2$. Notice that $|(1/2 + i/2)| = \sqrt{1/4 + 1/4} = \sqrt{2}/2 < 1$, so that all eigenvalues have absolute value < 1. Therefore $T^k \to 0$ as $k \to \infty$. Equivalently, every linear iterative system converges to 0 as well, so $u^{(k)} \to 0$ as $k \to \infty$ no matter what $u^{(0)}$ is.