Textbook: 1.1.1c, 1.2.1, 1.2.4c, 1.2.5ab, 1.2.7abcd, 1.3.1c

Extra Problem #1: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Find all real 2×2 matrices that commute with A.

Solution (1.2.1). (a) 3×4, (b) 7, (c) 6, (d) $\begin{pmatrix} -2 & 0 & 1 & 3 \end{pmatrix}$, (e) $\begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$

Solution (1.3.1c). We form the augmented matrix out of the system and row reduce.

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ -1 & 3 & 3 & -2 \\ 4 & -3 & 0 & 7 \end{array}\right)$$

$$r_2' = r_1 + 2r_2 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 4 & -3 & 0 & 7 \end{array}\right)$$

$$r_3' = \tfrac{-1}{5}(-2r_1 + r_3) \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 0 & 1 & 4/5 & -1/5 \end{array}\right)$$

$$r_3' = r_2 - 7r_3 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 0 & 0 & 12/5 & 2/5 \end{array}\right)$$

$$r_3' = \tfrac{5}{12} r_3 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 7 & 8 & -1 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_2' = \tfrac{1}{7} r_2 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 1 & 8/7 & -1/7 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_2' = \tfrac{-8}{7} r_3 + r_2 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_1' = -2r_3 + r_1 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 0 & 8/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_1' = -r_2 + r_1 \qquad \left(\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

$$r_1' = \tfrac{1}{2} r_1 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/6 \end{array}\right)$$

Thus $u = 3/2$, $v = -1/3$, and $w = 1/6$.

Solution (Extra Problem). Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If B commutes with A, then $AB = BA$. Expanding this equation, we obtain

$$\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}.$$

By equating the corresponding entries, we see that $a = d$ and $c = 0$. So therefore if B commutes with this particular matrix A, then it is of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Conversely it is easy to check that all matrices of this form commute with A.