7.5.1ab (don't choose, do (a) and (b)), 8.2.1ae, 8.2.2a, 8.2.12, 8.2.14, 8.2.17, 8.2.20, 8.2.35, 8.3.13ad, 8.3.14

Hint for 8.2.35: Consider $\operatorname{img}(P)$ and $\ker(P)$.

Solution (8.2.35). Let P be an idempotent matrix, so that $P^2 = P$. We saw in 8.2.20 that if λ is an eigenvector for A with eigenvector v, then λ^2 is an eigenvector for A^2 also with eigenvector v. Applying this to P, we see that on the one hand $P^2v = \lambda^2 v$, but on the other hand

$$P^2v = Pv = \lambda v.$$

Therefore $\lambda^2 v = \lambda v$. Since $v \neq 0$, then $\lambda^2 = \lambda$. Finally, we can conclude that an eigenvalue of an idempotent matrix is either $\lambda = 0$ or $\lambda = 1$.

To find the eigenvectors, we claim that $V_0 = \ker P$ as usual, and $V_1 = \operatorname{img}(P)$. The first claim we saw in class, the eigenvectors for the eigenvalue $\lambda = 0$ is always the kernel.

To show that $V_1 = \operatorname{img}(P)$, let $w \in V_1$, i.e. $\lambda = 1$ and $Pw = w$. Then by this very equation, $w = Pw$, we see that $w \in \operatorname{img}(P)$. It's the image of itself. So $V_1 \subseteq \operatorname{img}(P)$. Conversely let $w \in \operatorname{img}(P)$. Then we show that w is an eigenvector with $\lambda = 1$. If $w = Pu$ for some u, then

$$Pw = P(Pu) = P^2u = Pu = w.$$

Therefore w is an eigenvector for P with $\lambda = 1$. This shows that $V_1 \subseteq \operatorname{img}(P)$. These two subset relations show that $V_1 = \operatorname{img}(P)$.

Thus $V_1 = \operatorname{img}(P)$ and $V_0 = \ker(P)$. Since the only eigenvalues were $\lambda = 0, 1$, then we have completed the calculation. Notice that by rank-nullity, $\dim V_0 + \dim V_1 = \dim R^n$. Therefore we have a basis of eigenvectors by finding a basis of the kernel and the image, so idempotent matrices are diagonalizable.

Solution (8.2.14). (a) The matrix

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

has characteristic polynomial

$$p_A(\lambda) = -\lambda^3 + 3\lambda^2 + 13\lambda - 15.$$

The integer possibilities for the eigenvalues are $\pm 1, \pm 3, \pm 5$. Polynomial long division shows that the polynomial factors as

$$p_A(\lambda) = -(\lambda - 1)(\lambda + 3)(\lambda - 5)$$

so that the eigenvalues are $\lambda = 1, -3, 5$. The corresponding eigenvectors are

$$\begin{aligned} V_1 &= \ker(A - 1I) = \operatorname{span}(-2, -3, 3) \\ V_{-3} &= \ker(A + 3I) = \operatorname{span}(2, -3, 1) \\ V_5 &= \ker(A - 5I) = \operatorname{span}(2, 1, 1) \end{aligned}$$

(b) Indeed $\operatorname{tr}(A) = 1 + (-1) + 3 = 3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1 - 3 + 5 = 3$. This is the same as the coefficient of λ^2 in the characteristic polynomial.

(c) Indeed $\det A = -15$ and $\lambda_1\lambda_2\lambda_3 = 1(-3)5 = -15$. This is the constant term in the characteristic polynomial.