Textbook: 8.4.2ac, 8.5.1ad, 8.5.14ad, 8.5.31ac, 8.6.1ab, 8.6.7acf, 8.2.23, 8.6.17

Note: Not a typo, 8.6.17 relies on 8.2.23. It might be good to do them together!

Hint for 8.6.17: Consider

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hint for 8.6.7f: There's a difficulty in this problem, which was not addressed in lecture. This matrix has 2 eigenvectors $v_1 = (0,1,1,0)$ and $v_2 = (1,0,0,0)$, so you need to make a Jordan chain with v_3, v_4. Trying to make a chain off of $(0,1,1,0)$ immediately yields not results. So you might turn towards $v_2 = (1,0,0,0)$. This does work but not as easily as you might expect.

Here's why. When creating a Jordan chain of length 3 or bigger, you need some generalized eigenvectors w_1, w_2, w_3 such that

$$(A - \lambda I)w_1 = 0 \quad (A - \lambda I)w_2 = w_1 \quad (A - \lambda I)w_3 = w_2.$$

Notice the properties of w_2. Since $(A - \lambda I)w_3 = w_2$, then $w_2 \in \operatorname{img}(A - \lambda I)$ AND $(A - \lambda I)w_2 = w_1$. Most of the time there is an obvious choice for w_2. But sometimes there is not. In general you have to find all solutions to $(A - \lambda I)x = w_1$ that also intersect that with $\operatorname{img}(A - \lambda I)$ to make sure that $x = w_2$ works. If $v_2 = (1,0,0,0)$,the row reducing the equation

$$(A - 2I)v_3 = (1,0,0,0)$$

looks like

$$v_3 = x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

And you might say, "Oh I can pick $v_3 = (0,-1,0,0)$!" but it turns out that $(0,-1,0,0) \notin \operatorname{img}(A - 2I)$ so there can be no v_4. You need to find x, z such that

$$x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{img}(A - 2I)$$

so that you can find v_4. This isn't super hard but kind of a pain in the butt. In this case you can find that $x = -1$ and $z = 1/2$ so that

$$v_3 = \frac{-1}{2} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

is the correct choice for v_3. I'll let you find v_4.

Solution (8.6.7c). This is an upper triangular matrix, so the eigenvalues are $\lambda = 1$ three times. But if we compute the eigenspace

$$\ker(A - 1I) = \ker \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

we find that $V_1 = \text{span}(1,0,0)$. Therefore we need a Jordan chain of length 3 to make enough generalized eigenvectors. Call this chain v_1, v_2, v_3, where $v_1 = (1,0,0)$.

To find v_2, we solve $(A - 1I)v_2 = (1,0,0)$. Doing row reduction reveals that $v_2 = x(1,0,0) + (0,1,0)$, so we can choose that $v_2 = (0,1,0)$. To find v_3, we can solve $(A - 1I)v_3 = (0,1,0)$, so that $v_3 = (0,-1,1)$. Therefore the Jordan form is

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the change of basis matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution (8.2.23). We show that AB and BA have the same eigenvalues. There are 2 big cases; when $\lambda = 0$ and when $\lambda \neq 0$. We'll do $\lambda \neq 0$ first.

Let λ be an eigenvalue for AB, so that $v \neq 0$ and $ABv = \lambda v$. Assume that $\lambda \neq 0$. Then consider the vector Bv. Note that $Bv \neq 0$, since if $Bv = 0$, then $ABv = 0 \neq \lambda v$. Now we show that Bv is an eigenvector for BA with eigenvalue λ!

$$(BA)(Bv) = B(ABv) = B(\lambda v) = \lambda(Bv).$$

One can make the reverse argument to show that an eigenvalue $\mu \neq 0$ of BA must be an eigenvalue for AB with eigenvector Av. Therefore AB and BA have the same nonzero eigenvalues.

Now we just have to show that if one of AB or BA has an eigenvector $\lambda = 0$, so does the other. Assume that AB has eigenvalue $\lambda = 0$, i.e. the exists a $v \neq 0$ such that $ABv = 0$. We can make almost a similar argument, except that $Bv = 0$ is possible now. But if $Bv \neq 0$, then $BA(Bv) = (BA)(Bv) = 0$ so that Bv is an eigenvector with eigenvalue 0. But what happens if $Bv = 0$. Then it can't be an eigenvector. So, in that case there are 2 options.

Either A^{-1} exists or it doesn't. If it doesn't, then $\ker A \neq 0$, so that $0 \neq w \in \ker A$ and w is an eigenvector for BA with eigenvalue $\lambda = 0$. In that case we are done; 0 is an eigenvalue of both AB and BA.

If A^{-1} does exist and $Bv = 0$, then $BAw = 0$ can be solved by $w = A^{-1}v \neq 0$. In this case

$$BAw = BA(A^{-1}v) = B(AA^{-1})v = Bv = 0.$$

Then $\lambda = 0$ is also an eigenvalue for BA, with eigenvector $A^{-1}v$. In this case $\lambda = 0$ is an eigenvector for AB and BA and we are done.

Solution (8.6.17). We can show that AB and BA, while they have the same eigenvalues, do NOT have the same Jordan forms.

As a counterexample, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $AB = 0$, the zero matrix. And $BA = A$ it turns out. But the zero matrix has a triple eigenvalues $\lambda = 0$ with enough eigenvector $\dim V_0 = \dim \ker AB = 3$. But since $BA = A$, we can find the Jordan form of A. This matrix A has triple eigenvalues $\lambda = 0$, but the eigenspace V_0 has dimension 2. Therefore they have different Jordan forms, since one has enough eigenvectors, while the other does not.