Textbook: 5.2.1, 5.3.4b, 5.3.13, 5.4.1b, 5.5.1b, 8.7.2a, 8.7.13

Solution (5.2.1). Let

$$f(x, y, z) = x^2 + 2xy + 3y^2 + 2yz + z^2 - 2x + 3z + 2.$$

To find the minimum we write the function as $q(x) = x^T Kx - 2x^T f + c$ so that the minimum value occurs at $x^* = K^{-1}f$ and $q(x^*) = c - f^T x^*$ whenever K is positive definite.

Writing quadratic terms as a matrix and factoring a -2 out of the linear terms, we get that

$$K = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad f = \begin{pmatrix} 1 \\ 0 \\ -3/2 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Indeed K is positive definite since it's eigenvalues are $\lambda = 1, 2 \pm \sqrt{3} > 0$. Then the minimum occurs at

$$x^* = K^{-1}f = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$$

so the minimum value is

$$f(x^*) = 2 - f^T x^* = -3/2.$$

Solution (5.3.13). This problem asks to find the vector $w^* \in \operatorname{span}((0, 0, 1, 1), (2, 1, 1, -1))$ such that $\|w^* - b\|^2$ is minimized, where $b = (0, 3, 1, 2)$. This span is equal to the image of the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Therefore, the coordinates of w^* within $\operatorname{img}(A)$ is $x^* = (A^T A)^{-1}A^T b$, so that $w^* = Ax^* = A(A^T A)^{-1}A^T b$. We just have to calculate this out. We obtain

$$w^* = \frac{1}{14}\begin{pmatrix} 8 \\ 4 \\ 25 \\ 17 \end{pmatrix}.$$

Alternatively, notice that the given basis for $\operatorname{img}(A)$ is orthogonal, we could have computed, the projection of b onto the image of A. We would have gotten the same w^*.