Textbook: 9.1.14ac, 9.2.2ac, 9.2.3ab, 9.2.23bc, 9.3.2, 9.3.5, 2.6.6

Extra Problem: Consider the graph given by the adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Compute the Euler characteristic of this graph, using any method you wish.

*N*ote on 9.2.2: A matrix A being convergent means $A^k \to 0$ as $k \to \infty$.

Solution (9.2.3b). We check whether

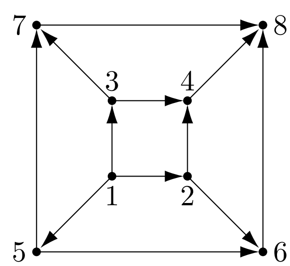
$$T = \frac{1}{12}\begin{pmatrix} 6 & 9 \\ 8 & 4 \end{pmatrix}$$

defines a linear iterative system with asymptotically stable zero solution. It suffices to determine whether the eigenvalues have absolute value less than 1. Calculating the eigenvalues of T, we obtain

$$\lambda = \frac{1}{12}(5 \pm \sqrt{73}).$$

Calculating out the terms, turns out that $5 + \sqrt{73} > 12$ so $|\lambda_1| > 1$. Therefore it defines a linear iterative system such that $u^* = 0$ is not a global asymptotically stable solution.

Solution (2.6.6). These matrices may change depending on how you label your vertices, but you should get 5 of the sides of the cube give independent circuits, but the last side is dependent on the other 5. I picked the inside 4 vertices to be 1,2,3,4 and the outside 4 to be 5,6,7,8, 1 and 5 are in the bottom left, 2 and 6 in the bottom right, 3 and 7 in the top left, and 4 and 8 in the top right. Here's a picture.



The incidence graph of the cubical graph is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Calculating the cokernel on the computer, the columns of the following matrix form a basis of the cokernel.

$$\begin{pmatrix} -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the rows of the above matrix correspond to edges of the graph, the column vectors correspond circuits 13421, 15621, 15731, 26842, 1378421. The first 4 circuits are just 4 sides to the cube, but last one is some other thing. But notice that $v_5 - v_1$ corresponds to the cycle 37843, which is the 5th side of the cube, and it's independent from the first 4. Therefore, the cokernel of A has a basis given by 5 circuits corresponding the 5 sides of the cube. The last side is dependent on the other 5.

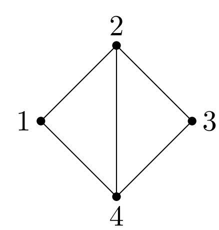
(b) The sixth side of the cube is 57865. We can write this as a linear combination of the other 5 sides as

$$57865 = 57315 + 78437 - 68426 - 56215 + 12431.$$

Pictorially, the "+" terms correspond to doing the circuit in the forward direction, and "-" refer to the opposite direction. The arrows in opposite directions cancel, and so that linear combination of circuits make the final side of the cube.

Just for kicks, the linear combination $56215 - 56731 = 5731265$ corresponds to a circuit which is two sides of the cube. And the linear combination $15621 - 13421 = 1562431$ also corresponds to subtracting two adjacent sides, the arrow in between them cancelling, and you get a circuit that goes around two sides.

Solution (Extra Problem). This adjacency matrix refers to the following undirected graph.



We saw that this graph has two independent circuits, 1241 and 2342. Therefore the Euler characteristic is $1 -$ number of independent circuits $= 1 - 2 = -1$. Alternatively, we could have calculated

$$\chi(G) = \text{number of vertices} - \text{number of edges} = 4 - 5 = -1.$$

Even more alternatively, the number of independent circuits is the dimension of the cokernel of the incidence matrix. We direct the graph arbitrarily. I picked 12, 23, 34, 41, 24. The transpose of the incidence matrix is

$$A_{\text{inc}}^T = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}.$$

The kernel of this is generated by the columns

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The cokernel is 2 dimensional, so the Euler characteristic is $\chi(G) = 1 - 2 = -1$.

Even more alternatively, we can consider the series of linear transformations

$$0 \longrightarrow \operatorname{span}(e_{12}, e_{23}, e_{34}, e_{41}, e_{24}) \longrightarrow \operatorname{span}(v_1, v_2, v_3, v_4) \longrightarrow 0$$

where the middle map is the boundary operation ∂. For example $\partial(e_{12}) = v_2 - v_1$. Then

$$\chi(G) = -\dim\ker(\partial) + \text{number of vertices} - \dim\operatorname{img}(\partial) = -2 + 4 - 3 = -1.$$