Textbook: 1.4.21e, 1.8.1be, 1.8.4, 1.8.23c, 1.8.24, 1.9.1cf, 1.9.5, 2.1.1, 2.1.3, 2.1.12, 2.1.13, 2.2.1

Extra Challenge Problem (Optional): Show that permutation formula for the determinant satisfies all 4 axioms for the determinant in Theorem 1.50.

Solution (1.8.1e). The augmented matrix for the system is

$$\left(\begin{array}{cccc|c} 1 & -2 & 2 & -1 & 3 \\ 3 & 1 & 6 & 11 & 16 \\ 2 & -1 & 4 & 1 & 9 \end{array}\right)$$

and row reduction yields

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The number of leading 1s is less than the number of columns, so that there is on free variable and an infinite number of solutions. The 3rd column has no leading 1, or no pivot, so z is a free variable. We know from the reduced matrix that $y = 1$, $w = 0$, and $z = s$ Furthermore, $x = -2z + 5$. The solution in vector form is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The solution set has the shape of a line, starting at $(5, 1, 0, 0)$ and going in the direction of $(-2, 0, 1, 0)$. In more advanced language, the kernel of the original matrix A is dimension 1 since the rank is 3, so the solution set is an affine subspace of dimension 1 as well.

Solution (1.8.4). Given the augmented matrix

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & 2 & a & b \\ b & 2 & a & a \end{array}\right)$$

we can row reduce to

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & 2 & a-b & b-2 \\ b & 0 & b & 2-b+a \end{array}\right)$$

using the steps $r_2' = -r_1 + r_2$ and $r_3' = -r_2 + r_3$. This system has a nonunique solution iff there is a row of 0's in the left matrix. But there are only 2 ways to get a row of zeroes: if $b = 0$ or if $a = b$.

If $b = 0$, then the bottom row is $\begin{pmatrix} 0 & 0 & 0 & 2+a \end{pmatrix}$. So the system has no solution if $a \neq -2$, and has an infinite number of solutions if $a = -2$.

If $a = b$, then the system reduces to

$$\left(\begin{array}{ccc|c} a & 0 & a & 2 \\ 0 & 2 & 0 & a-2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This system has an infinite number of solutions, since the z column is free.

Solution (1.8.24). From Theorem 1.47, a homogeneous system has trivial solution iff the matrix is nonsingular. However, an upper triangular matrix is nonsingular iff U has n pivots. But the only way an upper triangular matrix can have n pivots is if the diagonal entries are all nonzero. Therefore the system has trivial solution iff the diagonal entries are all nonzero.

Another proof is as follows. A matrix is nonsingular iff $\det U \neq 0$. But the determinant of an upper triangular matrix is the product of the diagonal entries (easy to check using permutation formula). Therefore the system will have nontrivial solution only a diagonal entry is 0 making the determinant 0.

Here is a more computational approach you can take to understand the situation. Assume that U has a nontrivial solution $v = (v_1 \dots v_n)^T \neq 0$. Then assume for contradiction that U has all nonzero entries on the diagonal, we know that $u_{nn}v_n = 0$. Since $u_{nn} \neq 0$, we must conclude that $v_n = 0$. Similarly, now we know that the previous row now is $u_{n-1,n-1}v_{n-1} = 0$. Again we conclude that $v_{n-1} = 0$. So by induction $v = 0$. But this is a contradiction, since $v \neq 0$. Thus one of the $u_{ii} = 0$. In other words, the back substitution algorithm tells us that the solution must be trivial of $u_i \neq 0$.

The converse is annoying with this method, so that's why the pivot and determinant proofs are easiest. We'll understand this result another way when we cover eigenvalues.

Solution (1.9.5). Let A and B be similar matrices, i.e. $B = S^{-1}AS$. Then

$$\det B = \det S^{-1}AS = \det S^{-1} \det A \det S = \frac{1}{\det S} \det A \det S = \det A.$$

Solution (2.1.1). We verify that $\mathbb{C}$ is a vector space over $\mathbb{R}$. Most of these follows just from the definition of complex addition and multiplication. The addition is commutative, for

$$(x + iy) + (u + iv) = (x + u) + i(y + v) = (u + x) + i(v + y) = (u + iv) + (x + iy).$$

Associativity follows similarly. The 0 element is just $0 = 0 + 0i \in \mathbb{C}$. The additive inverse is $-x - iy$. We can prove distributivity by noting

$$(c + d)(x + iy) = (c + d)x + i(c + d)y.$$

Similarly for associativity. The scalar unit is $1 = 1 + 0i$.

The vector space of complex numbers is basically just $\mathbb{R}^2$ except you write a vector like $x + iy$ instead of (x, y). The complex numbers can be multiplied together so they have extra information though!

Solution (2.1.12). Assume a vector space has two 0 vectors, call them 0 and $0'$. Then by definition of a 0 element, we know that

$$0' + 0 = 0'.$$

But since $0'$ is also a zero element, then $0 + 0' = 0$ also. Since both are equal to $0 + 0'$, we conclude that $0 = 0'$, and the zero element is unique.

Solution (Extra Problem). (i) Let A' be the matrix obtained from A by replacing r_i with $cr_i + r_j$. Given a permutation σ, denote σ^{-1} to be the backwards rearrangement (or the function inverse if viewing a permutation as a bijection). Then we can factor out each term of the form $cr_i + r_j$. For every permutation, the element's column will be $\sigma^{-1}(j)$.

$$\begin{aligned}\det A' &= \sum_\sigma \operatorname{sgn}\sigma \prod_k (A')_{\sigma(k),k} \\ &= \sum_\sigma (\operatorname{sgn}\sigma)(c(A)_{i,\sigma^{-1}(j)} + (A)_{j,\sigma^{-1}(j)}) \prod_{\sigma(k)\neq j} A_{\sigma(k),k} \\ &= \sum_\sigma (\operatorname{sgn}\sigma) c(A)_{i,\sigma^{-1}(j)} + \sum_\sigma (\operatorname{sgn}\sigma) \prod_k (A)_{\sigma(k),k} \\ &= c\left(\sum_\sigma (\operatorname{sgn}\sigma)(A)_{i,\sigma^{-1}(j)}\right) + \det A.\end{aligned}$$

So therefore it suffices to show $\sum_\sigma \operatorname{sgn}(\sigma)(A)_{i,\sigma^{-1}(j)} = 0$. We can organize this sum by permutations for which $\sigma^{-1}(j)$ is fixed. Namely the permutations that send $\sigma(k) = j$ is just a rearrangement of the $n-1$ other numbers. Thus

$$\sum_\sigma \operatorname{sgn}(\sigma)(A)_{i,\sigma^{-1}(j)} = \sum_{k=1}^n \sum_{\sigma|\sigma(k)=j} \operatorname{sgn}(\sigma)(A)_{i,k}$$

Therefore we can show that $\sum_{\sigma|\sigma(k)=j} \operatorname{sgn}(\sigma)(A)_{i,k} = 0$ for each k individually. Now we claim that for each σ such that $\sigma(k) = j$ for a fixed k, that half of these σ have positive sign and half of these have negative sign, so that the sum cancels to 0 for each k. Given a permutation σ, pick indices α and β such that $\alpha, \beta \neq k$. Then $(\alpha\beta)\sigma$ also sends $k \mapsto j$, but with opposite sign to σ. This gives a bijective association between positive and negative signed permutations that send k to j. Thus there are the same amount of permutations that have positive and negative sign. Thus all the terms cancel and the sum is 0. This completes the proof.

(ii) Let A be a matrix and let A' be the matrix with the ith and jth columns switched. Consider the permutation (ij). Given a permutation σ, denote $(ij)\sigma$ to be the permutation σ followed by (ij), so it is another permutation. Since we switched the i and j columns, then we claim that

$$(A')_{k,\sigma(k)} = (A)_{i,(ij)\sigma(k)}.$$

So the terms of the determinant of A' can be written in terms of that of A. It now just comes down to rearranging the sum properly. By definition,

$$\det A' = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \prod_k (A')_{k,\sigma(k)} = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \prod_k (A)_{i,(ij)\sigma(k)}$$

The product terms are almost the same, but how does $(ij)\sigma$ compare to just σ? This comes down to just a reindexing. Rename $\tau = (ij)\sigma$. Since we just add one transposition

$$\operatorname{sgn}\tau = \operatorname{sgn}(ij)\sigma = -\operatorname{sgn}\sigma.$$

Furthermore multiplying by (ij) just rearranges all the permutations, so the index set is the same. This is because we can just multiply by (ij) in the front. Switching i and j twice doesn't do anything!

$$(ij)\tau = (ij)(ij)\sigma = \sigma$$

(This is basically the same as renaming $n+1$ to be m in infinite sums. You can get back to the original $n+1$ just be subtracting 1, $n = m-1$. Here the (ij) plays the role of adding and subtracting 1.) So the index τ runs over the same permutations as σ, namely all of them. The only difference is that the sign is different. Therefore

$$\det A' = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \prod_k (A)_{i,(ij)\sigma(k)} = \sum_{\tau\in S_n} -\operatorname{sgn}\tau \prod_k (A)_{i,\tau(k)} = -\det A.$$

(iii) Now let A' be the matrix obtain by multiplying a row by a constant c, say row j. Then

$$\det A' = \sum_{\sigma} \operatorname{sgn} \sigma \prod_{i=1}^{n} (A')_{\sigma(i),i}$$

and each term in the product contains one entry multiplied by c, since a permutation is a bijection. Thus

$$\det A' = \sum_{\sigma} \operatorname{sgn} \sigma c \prod_{i=1}^{n} (A')_{\sigma(i),i} = c \sum_{\sigma} \operatorname{sgn} \sigma \prod_{i=1}^{n} (A)_{\sigma(i),i} = c \det A.$$

(iv) Finally, let U be an upper triangular matrix. We claim that the only permutation such that $\prod_{i=1}^{n} (U)_{\sigma(i),i} \neq 0$ is for the identity permutation. Indeed we can show for all nonidentity permutation, there exists an index j such that $\sigma(j) > j$. Then $(U)_{\sigma(j),j} = 0$ since U is upper triangular, and the product is 0. Now if σ is not the identity, then there is an i such that $\sigma(i) \neq i$. Consider the cycle generated by i, namely

$$i \mapsto \sigma(i) \mapsto \sigma(\sigma(i)) \mapsto \cdots \mapsto i.$$

Since $\{1, \ldots, n\}$ is a finite set, this cycle will eventually loop back to i. Since this cycle can't be decreasing for all steps, then there's a step in the cycle where the number goes back up. If we call this step j, then $\sigma(j) > j$. Then as stated above, the term $\prod_{i=1}^{n} (U)_{\sigma(i),i} = 0$. The only nonzero summand is when σ is the identity, and this is the product of the diagonal elements. Thus

$$\det U = \sum_{\sigma} \operatorname{sgn} \sigma \prod_{i=1}^{n} (U)_{\sigma(i),i} = \prod_{i=1}^{n} (U)_{i,i} + 0 = (U)_{1,1} (U)_{2,2} \ldots (U)_{n,n}.$$