Textbook: 2.4.21, 2.5.38, 2.5.39, 3.1.2abcd, 3.1.12, 3.1.21ab, 3.1.23abd, 3.1.26

Solution (2.4.21). Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$ and A an $n \times n$ nonsingular matrix. We show that $\{Av_1, \ldots, Av_n\}$ is also a basis of $\mathbb{R}^n$. For future reference, let $P = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$ be the $n \times n$ matrix with columns v_i. It is nonsingular since the v_i form a basis.

Since our potential basis has n vectors, and $\dim(\mathbb{R}^n) = n$, by Theorem 2.31d, it suffices to show that $\{Av_1, \ldots, Av_n\}$ is an independent set. Consider a linear combination

$$c_1 Av_1 + \cdots + c_n Av_n = 0.$$

Expanding this linear combination as vectors in $\mathbb{R}^n$, we get that

$$\begin{pmatrix} Av_1 & \ldots & Av_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But by the column formula for matrix multiplication, this is the linear system

$$APc = 0$$

where $c = (c_1, \ldots, c_n)^T$. Since both A and P are nonsingular, then AP is nonsingular and invertible. Thus there is a unique solution to this linear system, namely $c = 0$. Therefore the coefficients of the linear combination above were zero to begin with so $\{Av_1, \ldots, Av_n\}$ is independent, and by Theorem 2.31d, it is a basis.

Solution (2.5.38). Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $B \in \mathcal{M}_{p\times m}(\mathbb{R})$. Then we show that $\ker A \subseteq \ker BA$.

It suffices to show that if $v \in \ker A$, then $v \in \ker BA$ also. Writing out the defintions, we know that $Av = 0$. But then

$$BAv = B0 = 0.$$

This equation means that $v \in \ker BA$ and the proof is complete.

Solution (2.5.39). Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $B \in \mathcal{M}_{n\times p}(\mathbb{R})$. Then we show that $\operatorname{img} AB \subseteq \operatorname{img} A$.

First we claim that $\operatorname{img} AB \subseteq \{v = Aw \mid w \in \operatorname{img} B\}$, namely that all vectors v in the image of AB are of the form Aw, where w is in the image of B. In particular, since $v \in \operatorname{img} AB$, then $v = ABu$, where $u \in \mathbb{R}^p$. If we let $Bu = w$, then $v = Aw$. Furthermore $w \in \operatorname{img} B$ since $w = Bu$. Then we have shown the first step, that

$$\operatorname{img} AB \subseteq \{v = Aw \mid w \in \operatorname{img} B\}.$$

(In fact, these sets are equal, but we won't need equality.) Now, we know that $\operatorname{img} A = \{v = Aw \mid w \in \mathbb{R}^n\}$, which is the above set without the condition that $w \in \operatorname{img} B$. Therefore $\{v = Aw \mid w \in \operatorname{img} B\} \subseteq \operatorname{img} A$. Stringing these inequalities together, we see that

$$\operatorname{img} AB \subseteq \{v = Aw \mid w \in \operatorname{img} B\} \subseteq \operatorname{img} A$$

as desired.

Solution (3.1.2). (a) It is an inner product (b) Not an inner product since it is not positive, $\langle (1,0), (1,0)\rangle = 0$ even though $(1,0) \neq (0,0)$. (c) Similarly is not positive, since $\langle (1,-1), (1,-1)\rangle = 0 \cdot 0 = 0$. (d) Not an inner product since it is not bilinear

Solution (3.1.12). This can be solved by expanding out the right side and using bilinearity and symmetry.

$$\begin{aligned}||x+y||^2 - ||x-y||^2 &= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &\quad - \langle x, x\rangle + 2\langle x, y\rangle - \langle y, y\rangle \\ &= 4\langle x, y\rangle.\end{aligned}$$

Solution (3.1.23). (a) It is an inner product. I'll prove positivity since that is the hardest part. Note that $w(x) = e^{-x} > 0$ on $[-1, 1]$ and for any f, we have that $f(x)^2 \geq 0$. Therefore

$$\langle f, f\rangle = \int_{-1}^{1} f(x)^2 e^{-x}\, dx \geq 0.$$

But $\geq$ is not good enough. We have to show that $\langle f, f\rangle > 0$ if $f \neq 0$, which is a little harder. If $f(x) \neq 0$, then since it is continuous, there is an interval $I = (a, b) \subseteq [-1, 1]$ such that $f(x)^2 > 0$ on I. Otherwise f would just be the zero function or not continuous.

(For those of you with analysis background, since $f(x) \neq 0$, there exists an x_0 such that $f(x_0) \neq 0$. Let $\epsilon = f(x_0)$. Since f is continuous, there exists a δ such that $|f(x) - f(x_0)| < f(x_0)$ on $(x_0 - \delta, x_0 + \delta)$. In particular, $f(x) \neq 0$ in a neighborhood of x_0. Then let $I = (x_0 - \delta, x_0 + \delta)$ and $f(x)^2 > 0$ on that open interval.)

Anyway, we know that $f(x)^2 > 0$ (strict inequality is the key here!) on some smaller interval $I = [a, b]$ and consequentially $f(x)^2 e^{-x} > 0$ on the smaller interval also. Then if $f \neq 0$, we have that

$$\langle f, f\rangle = \int_{-1}^{1} f(x)^2 e^{-x}\, dx \geq \int_{a}^{b} f(x)^2 e^{-x}\, dx > 0.$$

Therefore this pairing is positive. (b) It is not an inner product since it is not positive. Let $f(x) = x$, then $\langle f, f\rangle = \int_{-1}^{1} x^3\, dx = 0$. (d) It is an inner product. Proof of positivity is the same as before, you just have to be a little bit careful since $x^2 = 0$ when $x = 0$ so you can't show that $f(x)^2 x^2 > 0$ like before. You have to make sure that your interval doesn't contain 0, but you can still do that since you can pick a subinterval of $[a, b]\backslash\{0\}$. Then on this even smaller inteval, which you can just rename back to $[a, b]$, you know that

$$\int_{-1}^{1} f(x)^2 x^2\, dx \geq \int_{a}^{b} f(x)^2 x^2\, dx > 0.$$

Just because $x^2 = 0$ exactly at $x = 0$, one single point won't affect the integral, it'll still be strictly positive.

Solution (3.1.26). Not true. A counterexample is $f(x) = x$ on $[a, b] = [0, 1]$. Then

$$||x||^2 = \int_{0}^{1} x^2\, dx = \frac{1}{3}.$$

But

$$||x^2|| = \sqrt{\int_{0}^{1} x^4\, dx} = \sqrt{\frac{1}{5}} \neq \frac{1}{3}.$$