Textbook: 3.2.2a, 3.2.10, 3.2.12ab, 3.2.16, 3.2.18, 3.2.25, 3.2.40, 3.3.10abcd, 3.3.20,abcd, 3.3.28abc, 3.3.35 (only part i), 3.3.39, 3.3.45ab, 3.3.47

Solution (3.2.10). (a) It is not an inner product because it is not symmetric nor is it positive. But it is bilinear.

(b) We can use the relation $\sin^2(\theta) = 1 - \cos^2(\theta)$ to solve this problem. First note that

$$(v \times w)^2 = ||v||^2||w||^2 - \langle v, w\rangle^2$$

by expanding both sides in coordinates. But then using the definition of cos,

$$(v \times w)^2 = ||v||^2||w||^2 - ||v||^2||w||^2\cos^2(\theta) = ||v||^2||w||^2\sin^2(\theta).$$

Taking the square root of both sides we see that $||v||||w||\sin(\theta) = \pm(v \times w)$. But is $v \times w$ positive or negative? We have to narrow it down to either $+$ or $-$. Turns out to be the $+$. To prove this, it suffices to show that $(v \times w) \geq 0$ when $\theta \in [0, \pi]$, which is the domain of cos.

Fix $||v||$ and $||w||$ and vary θ. Since $\sin(\theta) = 0$ only when v and w are parellel, i.e. when $\theta = 0, \pi$, it suffices to show that $v \times w > 0$ for any one $\theta \in (0, \pi)$ by the intermediate value theorem. We let $\theta = \pi/2$. Then $v = (v_1, v_2)$ and

$$w = \frac{||w||}{||v||}(-v_2, v_1)$$

which is 90 degree from v. Then

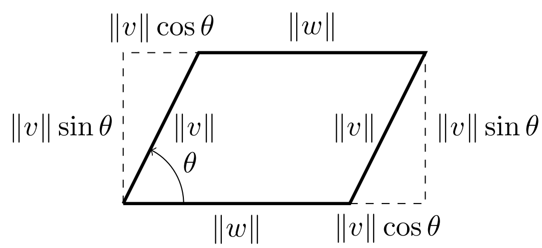
$$v \times w = \frac{||w||}{||v||}(v_1^2 + v_2^2) > 0$$

Thus we can take the positive root and

$$v \times w = ||v||||w||\sin(\theta).$$

(c) This follows from the formula. We know that $\sin(\theta) = 0$ if and only if $\theta = 0, \pi$ which is the case if and only if v and w are parellel.

(d) Consider the following diagram.



From here we can see that the area of the parallelogram is

$$A = (\|w\| + \|v\|\cos\theta)\,\|v\|\sin\theta - 2\left(\frac{1}{2}\,\|v\|\cos\theta\,\|v\|\sin\theta\right) = \|w\|\,\|v\|\sin\theta = |v \times w|$$

as desired. You can also see that $A = \left|\det\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}\right|$ which is how you generalize this to $\mathbb{R}^n$. You can only take a cross product in $\mathbb{R}^3$, if you remember from multi.

Solution (3.2.16). This is a row reduction problem. The vectors orthogonal $(1, 2, 3)$ and $(-2, 0, 1)$ are exactly the kernel of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{pmatrix}.$$

Row reduction tells you that the kernel is the span of the vector $(2, -7, 4)$.

Solution (3.2.25). We can do one better. Let $W \subseteq V$ be a subspace. Let $W^\perp$ be the set of all vectors orthogonal to all vectors of W. We can show that $W^\perp$ is a subspace. We have to show it satisfies the subspace axioms.

First $W^\perp \neq \emptyset$ since $0 \in W^\perp$. For all $w \in W$, we know that $\langle 0, w \rangle = 0$. The zero vector is orthogonal to everything.

Second, let $u, v \in W^\perp$. We show that $u + v \in W^\perp$ also. By bilinearity,

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = 0 + 0 = 0.$$

Therefore $u + v$ is orthogonal to all $w \in W$ as well.

Finally, $\langle cv, w \rangle = c\langle v, w \rangle = 0$ as well. So $W^\perp$ is a subspace. In particular, when $W = \text{span}(v)$, then $W^\perp$ is all the vectors perpendicular to v, like the problem asks.

Solution (3.2.40). It's true!

$$||w|| = || - v + (v + w)|| \leq || - v|| + ||v + w|| = ||v|| + ||v + w||$$

Solution (3.3.10c). We show that $||v|| = 2|v_1| + |v_2|$ is a norm on $\mathbb{R}^2$. First, it is positive by definition, since absolute values are always positive. Second, it is homogeneous since

$$||cv|| = 2|cv_1| + |cv_2| = |c|(2|v_1| + |v_2|) = |c| \cdot ||v||.$$

Finally the triangle inequality follows from the triangle inequality for the absolute value.

$$\begin{aligned} ||v + w|| &= 2|v_1 + w_1| + |v_2 + w_2| \\ &\leq 2(|v_1| + |w_1|) + |v_2| + |w_2| \\ &= 2|v_1| + |v_2| + 2|w_1| + |w_2| \\ &= ||v|| + ||w|| \end{aligned}$$

Solution (3.3.35i). We have to show that

$$||v||_2 \leq ||v||_1 \leq \sqrt{n}||v||_2.$$

First, we know that by distributive property

$$||v||_1^2 = \left(\sum_{i=1}^{n} |v_i|\right)^2 = \sum_{i=1}^{n} |v_i|^2 + \sum_{i \neq j}^{n} |v_i v_j| \geq \sum_{i=1}^{n} |v_i|^2 = ||v||_2^2.$$

Taking square roots, this proves the first inequality. For the next inequality, we can do a clever trick using Cauchy-Schwartz.

$$\begin{aligned} ||v||_1 = \sum_{i=1}^{n} |v_i| &= (|v_1|, \ldots, |v_n|) \cdot (1, 1, \ldots, 1) \\ &\leq ||(|v_1|, \ldots, |v_n|)|| \cdot ||(1, \ldots, 1)|| = ||v||_2 \sqrt{n} \end{aligned}$$

Another way to show this inequality is to find

$$c = \min\{||u||_1 \mid ||u||_2 = 1\}$$

and

$$d = \max\{||u||_1 \mid ||u||_2 = 1\}.$$

The minimum value of the L^1 norm on the L^2 unit sphere is achieved at $u = (1, 0, \ldots, 0)$, so that $c = 1$. The maximum is achieved at $u = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$ so that $d = \sqrt{n}$. Therefore $||v||_2 \le ||v||_1 \le \sqrt{n}||v||_2$.

Solution (3.3.39). Since $||f||_\infty$ is the maximum value of $|f(x)|$ on the interval $[a, b]$, then we know that $f(x)^2 \le ||f||_\infty^2$ on the interval as well. We can integrate both sides of this inequality and get

$$||f||_2^2 = \int_a^b f(x)^2\,dx \le \int_a^b ||f||_\infty^2\,dx = (b - a)||f||_\infty^2.$$

Taking the square root of both sides shows that

$$||f||_2 \le \sqrt{b - a}||f||_\infty.$$

Solution. It's false. I picked some random A and S matrices and it gave me a counterexample.

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad B = S^{-1}AS = \frac{1}{2}\begin{pmatrix} 10 & 11 & 9 \\ 22 & 23 & 21 \\ -2 & -1 & -3 \end{pmatrix}.$$

Then $||A||_\infty = 7 + 8 + 9 = 24$ but $||B||_\infty = (22 + 23 + 21)/2 = 33$.