Textbook: 4.1.2ab, 4.1.15, 4.1.17, 4.1.27ac (feel free to use wolframalpha for these integrals if you want), 4.2.1a, 4.3.2a

Optional Challenge Problem: Let σ be a permutation of n objects with corresponding permutation matrix P_σ. Let σ^{-1} be the reverse permutation to σ. Or if you know that σ is a bijection

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

let σ^{-1} be the inverse bijection. Show that $P_\sigma^T = P_{\sigma^{-1}}$. (Hint: Look at Exercise 4.3.15).

Solution (4.1.15). Let V be an inner product space. We show that $\|v\|^2 + \|w\|^2 = \|w + v\|^2$ iff v is perpendicular to w. This is the Pythagorean theorem in a general inner product space since when v and w are at a right angle, v, w, and $v + w$ form a right triangle.

First, let the Pythagorean formula hold. Then

$$\|v\|^2 + \|w\|^2 = \|v + W\|^2 = \langle v + w, v + w\rangle = \|v\|^2 + \|w\|^2 + 2\langle v, w\rangle.$$

By cancelling we see that $\langle v, w\rangle = 0$, so they are orthogonal. Conversely, if $\langle v, w\rangle = 0$, it is easy to see that $\|v + w\|^2 = \|v\|^2 + \|w\|^2$ by simplification.

Solution (4.1.17). Given nonzero mutually orthogonal vectors $v_1, \ldots, v_k$, their Gram matrix is $K \in M_{k\times k}(\mathbb{R})$ such that $(K)_{ij} = \langle v_i, v_j\rangle$. Since the v_i are mutually orthogonal, then if $i \neq j$, $(K)_{ij} = 0$, and on the diagonal $(K)_{ii} = \|v_i\|^2$. Then K is a diagonal matrix with $\|v_i\|^2$ on the diagonal. It is nonsingular since $v_i \neq 0$ so its diagonal are strictly positive, or nonzero at least. Therefore K has the full amount of pivots, and it is nonsingular.

Solution (4.1.27a). Let $P^{(3)}$ be the vector space of polynomials with degree 3 or less as a subspace of $C^0[-1, 1]$ with the usual inner product. This is four dimensional since $1, t, t^2$, and t^3 form a basis. We show that $P_0 = 1$, $P_1 = t$, $P_2 = t^3 - 1/3$, and $P_3 = t^3 - 3t/5$ form an orthongonal basis.

To show they form a basis in the first place, note that $P_2 + (1/3)(P_0) = t^2$ and $P_3 + (3/5)(P_1) = t^3$. Therefore $1, t, t^2, t^3 \in \operatorname{span}(P_0, P_1, P_2, P_3)$. Since all the elements of an already known basis are in the span of the P_i, then the P_i span $P^{(3)}$. Since there are four of them and $\dim(P^{(3)}) = 4$, then they form a basis by Theorem 2.31.

To show this is an orthogonal basis, we show that $\langle P_i, P_j\rangle = 0$ for $i \neq j$. You can simplify these integrals since most of these are integrals of an odd function on a symmetric interval. The inner products $\langle P_0, P_2\rangle$ and $\langle P_1, P_3\rangle$ cancel nicely though.

$$\begin{aligned}
\langle P_0, P_1\rangle &= \int_{-1}^{1} t\, dt = \frac{1}{2} - \frac{1}{2} = 0 \\
\langle P_0, P_2\rangle &= \int_{-1}^{1} t^2 - \frac{1}{3}\, dt = \frac{2}{3} - \frac{2}{3} = 0 \\
\langle P_0, P_3\rangle &= \int_{-1}^{1} t^3 - \frac{3}{5} t\, dt = \left(\frac{t^4}{4} - \frac{3t^2}{10}\right)_{-1}^{1} = \frac{-1}{10} - \frac{-1}{10} = 0 \\
\langle P_1, P_2\rangle &= \int_{-1}^{1} t\left(t^2 - \frac{1}{3}\right) dt = \left(\frac{t^4}{4} - \frac{t^2}{6}\right)_{-1}^{1} = 0 \\
\langle P_1, P_3\rangle &= \int_{-1}^{1} t\left(t^3 - \frac{3}{5} t\right) dt = \left(\frac{t^5}{5} - \frac{3t^3}{15}\right)_{-1}^{1} = 0 \\
\langle P_2, P_3\rangle &= \int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)\left(t^3 - \frac{3t}{5}\right) dt = \int_{-1}^{1} t^5 - \frac{14t^3}{15} - \frac{t}{5}\, dt = 0
\end{aligned}$$

Since these basis vectors are mutually orthogonal, then they form an orthogonal basis.

Solution (Optional Problem). Let $\sigma \in S_n$ with inverse σ^{-1}. Let P_σ be the corresponding permutation matrix to σ. We form σ by permuting the rows of I according to the permutation σ. Therefore P_σ has entries

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{else} \end{cases}$$

Therefore P_σ^T has entries

$$(P_\sigma^T)_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } i = \sigma^{-1}(j) \\ 0 & \text{else} \end{cases} = (P_{\sigma^{-1}})_{ij}.$$

Thus since P_σ^T and $P_{\sigma^{-1}}$ have the same entries, we see that $P_\sigma^T = P_{\sigma^{-1}}$.

Note that the columns of P_σ form an orthonormal basis, as they are just a rearrangement of the standard basis vectors by σ^{-1}. Therefore P_σ is orthogonal, and $P_\sigma^T = P_\sigma^{-1}$. Thus we have shown that $P_{\sigma^{-1}} = P_\sigma^{-1}$.