Textbook: 7.1.28, 7.1.32, 7.2.24ab, 7.4.32b, 7.4.41a

Solution (7.1.28). It's true. Let $W \subseteq \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ be the subset of transformations T such that $T(e_1) = 0$. If $S, T \in W$

$$(S + T)(e_1) = S(e_1) + T(e_1) = 0 + 0 = 0$$

so that $S + T \in W$. Furthermore $(cT)(e_1) = cT(e_1) = c \cdot 0 = 0$, so that $cT \in W$. Therefore W is a subspace.

Since it is a subspace, it must have a dimension. Recall that $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2) = \mathcal{M}_{2\times 2}(\mathbb{R})$, the set of 2×2 matrices. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$, then $Ae_1 = 0$. Writing this out in coordinates yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$A = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $W = \mathrm{span}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ and $\dim(W) = 2$.

Solution (7.4.32b). We solve the linear differential equation $u'' - 9u = x + \sin(x)$ using the superposition principle. First, we find a particular solution to $u'' - 9u = x$. We guess $u = cx$, since $u'' = 0$ in this case. Then $-9u = -9cx = x$, so $c = -1/9$. Therefore our particular solution is $u_1^* = -\frac{1}{9}x$.

To get the second particular solution, we solve $u'' - 9u = \sin(x)$. Here we guess $u = b\sin(x)$, so that $-b\sin(x) - 9b\sin(x) = \sin(x)$, and therefore $-b - 9b = 1$ which means $b = -1/10$. Therefore

$$u_2^* = -\frac{1}{10}\sin(x).$$

Finally, to get the homogeneous solution to $u'' - 9u = 0$, we guess $u = e^{rx}$, so that $(r^2 - 9)e^{rx} = 0$, and therefore $r^2 = 9$, so that $r = \pm 3$. The homogeneous solution is $z(x) = c_1 e^{3x} + c_2 e^{-3x}$, and so the general solution to the original equation is

$$\begin{aligned} u(x) &= z(x) + u_1^* + u_2^* && (1) \\ &= c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{9}x - \frac{1}{10}\sin(x) && (2) \end{aligned}$$