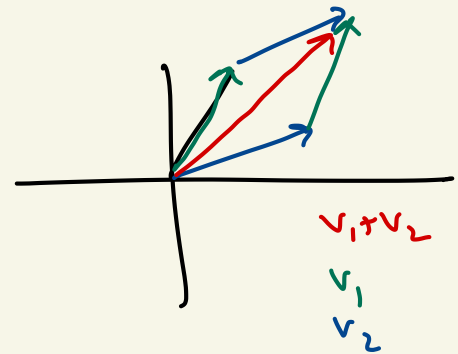
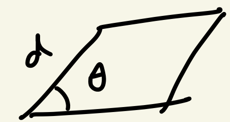
Chapter 3 Inner products, norms, positive definite matrices

$$\mathbb{R}^2 = \{ (x,y) \mid x,y \in \mathbb{R} \}$$



$$\vec{v}_1 + \vec{v} = (x_1 + x_2, y_1 + y_2)$$

Can I define/construct a notion of "distance" or "angle" on $\mathbb{R}^2$ w/ vector space properties?



"made a parallelogram"

Just the 7 vector space axioms aren't enough to do any meaningful geometry. To think about "distance" and "angle" we need an inner product!

Def Let V be a vector space $(\mathbb{R}^n, C^0[a,b], P^{(n)})$.

could be anything

An inner product on V is a "pairing"

Sort of a "product" of 2 vectors

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

vector space

need not be $\mathbb{R}^n$

input

$\vec{v}, \vec{w}$

output

$\langle \vec{v}, \vec{w} \rangle \in \mathbb{R}$

$V \times V$ just mean two vectors $\vec{v}, \vec{w}$

such that

1) Bilinearity : $\langle c\vec{v} + d\vec{w}, \vec{u} \rangle$

$$= c\langle \vec{v}, \vec{u} \rangle + d\langle \vec{w}, \vec{u} \rangle$$

$\langle \vec{v}, c\vec{w} + d\vec{u} \rangle$

$$= c\langle \vec{v}, \vec{w} \rangle + d\langle \vec{v}, \vec{u} \rangle$$

2) Symmetry : $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

3) Positivity : if $\vec{v} \neq 0$, $\langle \vec{v}, \vec{v} \rangle > 0$

if $\vec{v} = \vec{0}$, $\langle \vec{0}, \vec{0} \rangle = 0$

Ex let $V = \mathbb{R}^n = \{(x_1, \dots, x_n)\}$

Define $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

The dot product is an inner product. To see this,

we'll verify all 3 inner product properties for the dot product!

1) Bilinearity $(c\vec{v} + d\vec{w}) \cdot \vec{u}$

$$= \sum_{i=1}^{n} (cv_i + dw_i) u_i = \sum_{i=1}^{n} c(v_i u_i) + d(w_i u_i)$$

$$= \sum_{i=1}^{n} c(v_i u_i) + \sum_{i=1}^{n} d(w_i u_i)$$

$$= c \underbrace{\boxed{\sum_{i=1}^{n} v_i u_i}}_{v \cdot u} + d \underbrace{\boxed{\sum_{i=1}^{n} w_i u_i}}_{w \cdot u} = \underline{c(\vec{v} \cdot \vec{u}) + d(\vec{w} \cdot \vec{u})}$$

$$\vec{v} \cdot (c\vec{w} + d\vec{u}) = \sum_{i=1}^{n} v_i(cw_i + du_i)$$

$$= c\sum_{i=1}^{n} v_i w_i + d\sum_{i=1}^{n} v_i u_i$$

$$= c(\vec{v} \cdot \vec{w}) + d(\vec{v} \cdot \vec{u})$$

✓ We are <u>bilinear!</u>

2) <u>Symmetry</u>: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} \underbrace{x_i y_i}_{\text{plain old numbers}} = \sum_{i=1}^{n} y_i x_i = \vec{y} \cdot \vec{x}$

✓ It's symmetric!

3) positivity : We need to show that if $\vec{x} = (x_1, \dots, x_n) \neq \vec{0}$

then $\vec{x} \cdot \vec{x} > 0$.

$$\vec{x} \cdot \vec{x} = \sum_{i=1}^{n} x_i x_i = x_1 x_1 + x_2 x_2 + \dots + x_n x_n$$

$$= \sum_{i=1}^{n} x_i^2 .$$ One of the $x_i \neq 0$ since $\vec{x} \neq \vec{0}$

and squares are always positive,

$$\Rightarrow \vec{x} \cdot \vec{x} = \sum_{i=1}^{n} x_i^2 > 0$$

It's positive!

Furthermore $\vec{0} \cdot \vec{0} = \sum_{i=1}^{n} 0^2 = 0 + 0 + \dots + 0 = 0.$

The dot product is an inner product!

The dot product is the prototypical example of an inner product. But there are others.

<u>Ex</u> $V = \mathbb{R}^2 = \{(x_1, x_2)\}$.

Define $\langle \vec{x}, \vec{y} \rangle = 5x_1y_1 + 2x_2y_2$ weighted dot product or weighted inner product.

This an inner product.

Bilinearity! $\langle c\vec{v} + d\vec{w}, \vec{u} \rangle$

$= 5(cv_1 + dw_1)u_1 + 2(cv_2 + dw_2)u_2$

$= 5cv_1u_1 + 5dw_1u_1 + 2cv_2u_2 + 2dw_2u_2$

$= (5cv_1u_1 + 2cv_2u_2) + (5dw_1u_1 + 2dw_2u_2)$

$= c(5v_1u_1 + 2v_2u_2) + d(5w_1u_1 + 2w_2u_2)$

$= c\langle \vec{v}, \vec{u} \rangle + d\langle w, u\rangle$

Second linearity $\langle \vec{v}, c\vec{w} + d\vec{u} \rangle = c\langle \vec{v}, \vec{w}\rangle + d\langle \vec{v}, \vec{u}\rangle$

Similarly.

Symmetry: $\langle \vec{x}, \vec{y} \rangle = 5x_1y_1 + 2x_2y_2 = 5y_1x_1 + 2y_2x_2$

$= \langle \vec{y}, \vec{x} \rangle$

Positivity: If $\vec{x} \neq \vec{0}$, then

$\langle \vec{x}, \vec{x} \rangle = 5x_1^2 + 2x_2^2 > 0$.

$\langle \vec{0}, \vec{0} \rangle = 5 \cdot 0 \cdot 0 + 2 \cdot 0 \cdot 0 = 0$.

Therefore $\langle \vec{x}, \vec{y} \rangle = 5x_1y_1 + 2x_2y_2$ is an inner product!

Non Ex $\langle \vec{x}, \vec{y} \rangle = -5x_1y_1 + 2x_2y_2$ is not an inner product!

It fails 3) positivity axiom.

let $\vec{x} = (1,0)$

$$\langle (1,0), (1,0) \rangle = -5 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0 = -5$$

$\langle (1,0), (1,0) \rangle > 0$ should happen

$$\begin{array}{c} \langle (1,0), (1,0) \rangle \\ \| \\ -5 < 0. \end{array}$$

So it's not an inner product.

Key example let $V = C^0[a,b]$

$= $ all continuous functions on $[a,b]$.

Remember functions are vectors.

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$ is an inner product on $C^0[a,b]$

1) $$\langle cf + dg, h \rangle = \int_a^b \big(cf(x) + dg(x)\big)h(x)\,dx$$

$$= \int_a^b cf(x)h(x) + dg(x)h(x)\,dx$$

Calc I

$$= c\int_a^b f(x)h(x)\,dx + d\int_a^b g(x)h(x)\,dx$$

$$= c\langle f, h \rangle + d\langle g, h \rangle$$

$\langle f, cg + dh \rangle = c\langle f, g \rangle + d\langle f, h \rangle$ exactly the same.

2) Symmetry: $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

$$= \int_a^b g(x) f(x)\, dx = \langle g, f \rangle$$

3) If $f \neq 0$, then

$$\langle f, f \rangle = \int_a^b f(x)^2\, dx = \text{area under } f(x)^2 \text{ from } [a,b]$$

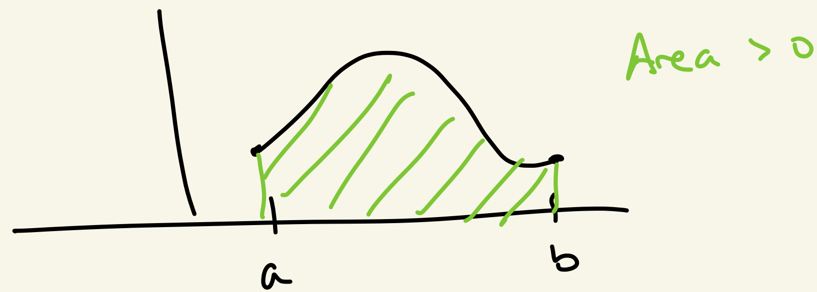
optional box

$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{else} \end{cases}$

$\int_{-1}^{1} f(x)^2\, dx = 0$!

f is not cts

$f(x)^2$ is a positive function



Since $f(x)^2 \geq 0$ and $f(x) \neq 0$ then $\int_a^b f(x)^2\, dx > 0$.

(continuity required)

Ex $\quad V = C^0[-1,1] \quad f(x) = x^2 \quad g(x) = 5.$

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

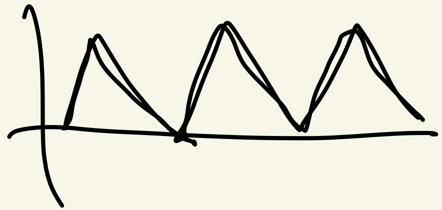
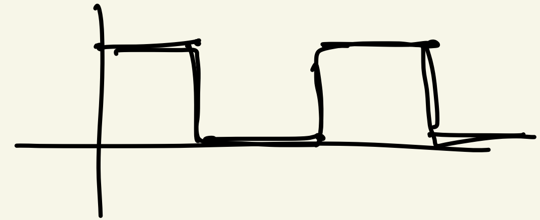
$$= \int_{-1}^{1} (x^2)\, 5\, dx = 5 \int_{-1}^{1} x^2\, dx$$

$$= 5 \left(\tfrac{1}{3} x^3 \right)_{-1}^{1} = \frac{5}{3}(1-(-1)) = \frac{10}{3}.$$

Why do inner product lead to angle and distance?

Look for HW4 right after this lecture. (due Friday).

Signals : periodic functions

functions $\longrightarrow$ no row reduce but inner products!

no matrices

Write a sawtooth or box as a linear combination

of $\sin(x)$ and $\cos(x)$

$\sin(2x)$ $\cos(2x)$

$\sin(3x)$ $\cos(3x)$

$\sin(4x)$ $\cos(4x)$

$\vdots$ $\vdots$

$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$ lets you do this calculation