Last time we defined inner products $\langle v, w \rangle$, input two vectors output a real number

1) Bilinearity

2) Symmetry $\langle v, w \rangle = \langle w, v \rangle$

3) Positivity $\langle v, v \rangle > 0$ if $v \neq 0$

- Dot product on $\mathbb{R}^n$
- weighted dot product $\langle \vec{x}, \vec{y} \rangle = 5x_1y_1 + 2x_2y_2$

  more generally $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} c\, x_i y_i$, where $c > 0$
- $\underline{L^2}$-inner product on C^0

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

While $\mathbb{R}^n$ has properties like matrices / row reduction and $C^0[a,b]$ doesn't, they both have inner products! Maybe we can learn something about $C^0[a,b]$.

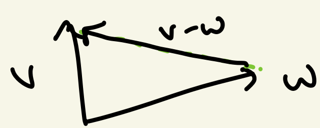
- Positivity $\langle v,v\rangle > 0 \quad v \neq 0$

Define the magnitude or <u>norm</u> of a vector to be $\|v\| = \sqrt{\langle v,v\rangle} > 0$

We can define the <u>distance</u> between v,w as

$$d(v,w) = \|v-w\| \geq 0$$

(distance is always positive!)

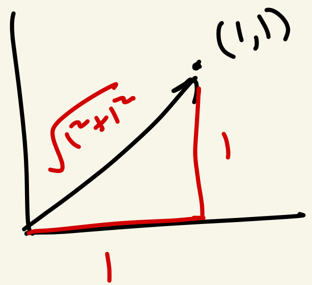


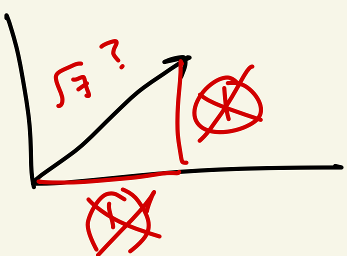
$\|v-w\|$ should be the "distance".

The magnitude depends on choice of inner product!

$$\|v\| = \sqrt{\langle v,v\rangle} > 0$$

Let $v = (1,1)$





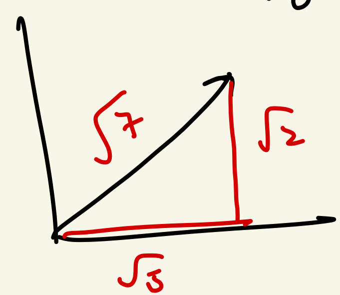
$(\mathbb{R}^2, v\cdot w)$

$$\|v\| = \sqrt{v\cdot v}$$
$$= \sqrt{(1,1)\cdot(1,1)}$$
$$= \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$

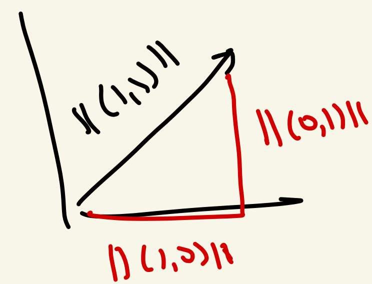
$(\mathbb{R}^2, 5x_1y_1 + 2x_2y_2)$

$$\|v\| = \sqrt{\langle v,v\rangle}$$
$$= \sqrt{5\cdot1\cdot1 + 2\cdot1\cdot1}$$
$$= \sqrt{5+2}$$
$$= \sqrt{7}$$

Pythagorean thm still holds!



$$(\sqrt{5})^2 + (\sqrt{2})^2 = (\sqrt{7})^2$$



Distance ✓

Angle ?

First, we need a fancy theorem.

<u>Thm</u> (Cauchy - Schwartz Inequality)

Let $v, w \in V$ w/ inner product $\langle -, - \rangle$.

Then $|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$.

Equality is true iff $\vec{v} = c\vec{w}$. → has similar proof

(v parallel to w)

<u>Pf</u> Let t be a constant in $\mathbb{R}$, $t \in \mathbb{R}$.

Consider $\vec{v} + t\vec{w}$. $\Rightarrow \underbrace{\|\vec{v} + t\vec{w}\|^2}_{\text{always positive}} \geq 0$. by $\langle \vec{x}, \vec{x} \rangle > 0$

$\|x\|^2 > 0$

$$0 \le \|\vec{v} + t\vec{w}\|^2 = \sqrt{\langle \vec{v} + t\vec{w}, \vec{v} + t\vec{w}\rangle}^{\,2} = \langle \vec{v} + t\vec{w}, \vec{v} + t\vec{w}\rangle$$

Bilinearity

$$= \langle v + tw, v\rangle + \langle v + tw, tw\rangle$$

$$= \langle \vec{v}, \vec{v}\rangle + \underbrace{\langle t\vec{w}, \vec{v}\rangle + \langle v, t\vec{w}\rangle}_{\text{symmetry}} + \langle t\vec{w}, t\vec{w}\rangle$$

$$= \|\vec{v}\|^2 + 2\langle \vec{v}, t\vec{w}\rangle + \|t\vec{w}\|^2$$

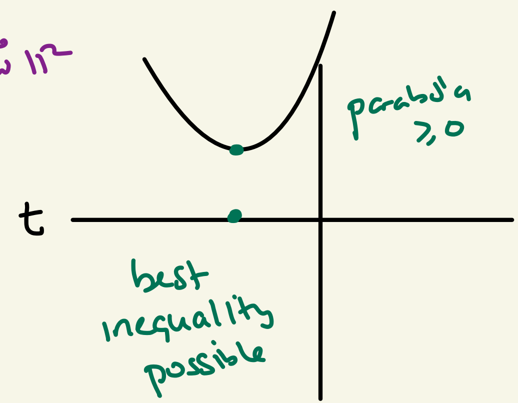
$$= \underbrace{\|\vec{v}\|^2}_{c} + \underbrace{2t\langle \vec{v}, \vec{w}\rangle}_{bt} + \underbrace{t^2\|\vec{w}\|^2}_{at^2} \ge 0 \quad * \quad t \text{ constant}$$

$c + bt + at^2$

$b = 2\langle \vec{v}, \vec{w}\rangle$ $\qquad a = \|\vec{w}\|^2$

Minimize in the variable t!

Calc I ....



Solve $\frac{d}{dt}\left(\cancel{\|\vec{v}\|^2}_{0} + 2t\langle\vec{v},\vec{w}\rangle + t^2\|\vec{w}\|^2\right) = 0$

linear

$$2\langle\vec{v},\vec{w}\rangle + 2t\|\vec{w}\|^2 = 0$$

$$t = \frac{-\langle\vec{v},\vec{w}\rangle}{\|\vec{w}\|^2}$$

Plug t back into the original inequality

$$\|\vec{v}\|^2 + 2t\langle\vec{v},\vec{w}\rangle + t^2\|\vec{w}\|^2 \geq 0$$

$$\Longrightarrow \|\vec{v}\|^2 + 2\left(\frac{-\langle\vec{v},\vec{w}\rangle}{\|\vec{w}\|^2}\right)\langle\vec{v},\vec{w}\rangle + \left(\frac{\langle v,w\rangle^2}{\|w\|^{\cancel{4}\,2}}\right)\|w\|^{\cancel{2}} \geq 0$$

$$\|\vec{v}\|^2 - \frac{\langle v,w\rangle^2}{\|w\|^2} \geq 0 \Longrightarrow \langle v,w\rangle^2 \leq \|v\|^2\|w\|^2$$

$$\Longrightarrow |\langle v,w\rangle| \leq \|v\|\cdot\|w\|$$

$\square$

C-S ineq. $|\langle v,w\rangle| \le \|v\| \cdot \|w\|$

let $v,w \neq 0$

$\|v\|, \|w\| \neq 0 \implies \dfrac{|\langle v,w\rangle|}{\|v\|\,\|w\|} \le 1$

$$\implies -1 \le \frac{\langle v,w\rangle}{\|v\|\,\|w\|} \le 1 .$$

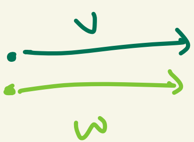
compute this in some familiar setting

let $V = \mathbb{R}^2$, w/ dot product.

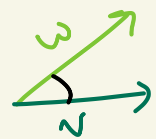
$v = (1,0)$

$w = (1,0)$

$$\frac{v \cdot w}{\|v\| \cdot \|w\|} = \frac{1}{1 \cdot 1} = 1 = \cos(0)$$



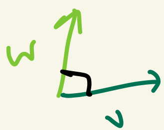
$\theta = 0^\circ$



$\theta = 45^\circ$
$= \pi/4$

$v = (1,0)$
$w = (1,1)$

$$\frac{v \cdot w}{\|v\| \cdot \|w\|} = \frac{1}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}} = \cos(\pi/4)$$



$\theta = \pi/2$

$v = (1,0)$
$w = (0,1)$

$$\frac{v \cdot w}{\|v\| \, \|w\|} = \frac{0}{1 \cdot 1} = 0 = \cos(\pi/2).$$

Let $v, w \neq 0$ in V w/ $\langle -,- \rangle$.

Then the angle θ between v, w is

$$\theta = \cos^{-1}\left(\frac{\langle v, w \rangle}{\|v\| \, \|w\|} \right)$$

$\cos^{-1}$ only exists on $[-1,1]$, so we need C-S to define this.

Let $V = C^0[a,b]$ = vector space of continuous functions on $[a,b]$.

$[a,b] = [0,1]$

$\vec{f} = x$

$\vec{g} = x^2$

$$d(\vec{f},\vec{g}) = \|f-g\| = \sqrt{\langle f-g, f-g\rangle}$$

Remember $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$ ....

$$\sqrt{\langle f-g, f-g\rangle} = \sqrt{\int_0^1 (x-x^2)(x-x^2)\,dx}$$

$$= \sqrt{\int_0^1 x^2 - 2x^3 + x^4\,dx} = \sqrt{\frac{1}{3} - \frac{1}{2} + \frac{1}{5}}$$

$$= \sqrt{\frac{1}{30}} = d(x,x^2)$$

$$\theta = \cos^{-1}\left(\frac{\langle x, x^2\rangle}{\|x\|\,\|x^2\|}\right) = \cos^{-1}\left(\frac{\int_0^1 x \cdot x^2\,dx}{\sqrt{\int_0^1 x^2\,dx}\,\sqrt{\int_0^1 x^4\,dx}}\right)$$

$$= \cos^{-1}\left(\frac{\frac{1}{4}}{\sqrt{\frac{1}{3}}\,\sqrt{\frac{1}{5}}}\right) = \cos^{-1}\left(\frac{\sqrt{\frac{1}{16}}}{\sqrt{\frac{1}{15}}}\right)$$

$$= \cos^{-1}\left(\sqrt{\frac{15}{16}}\right) = \text{something} \ldots$$

Bilinearity

$$\langle \vec{v}, \underline{c\vec{u} + d\vec{w}}\rangle = c\langle v,u\rangle + d\langle v,w\rangle$$

let $u = 0$ $\quad d = t$

- distribute over addition
- pull out constants

$$\langle v, tw\rangle = c\langle v,\cancel{0}\rangle + t\langle v,w\rangle$$

$$\langle v, tw\rangle = t\langle v,w\rangle$$