If $V = \mathbb{R}^2$ $\quad v = (v_1 v_2)$

$V = \mathbb{R}^3$ $\quad v = (v_1 v_2 v_3)$

depends on V and what inner product formula you have

$\langle v, cu + dw \rangle = c\langle v, u \rangle + d\langle v, w \rangle$ *

$\langle v, u+w \rangle = \langle v, u \rangle + \langle v, w \rangle$

$c, d = 1$

$\langle v, cu \rangle$

$= c\langle v, u \rangle$

$d, w = 0, \vec{0}$

Recall, given any inner product $\langle -,- \rangle$, we can define the distance between two vectors as

$$d(\vec{v},\vec{w}) = \|\vec{v}-\vec{w}\| = \sqrt{\langle v-w, v-w\rangle}$$

positivity axiom

Furthermore, if $\vec{v},\vec{w} \neq 0$, the angle between them is

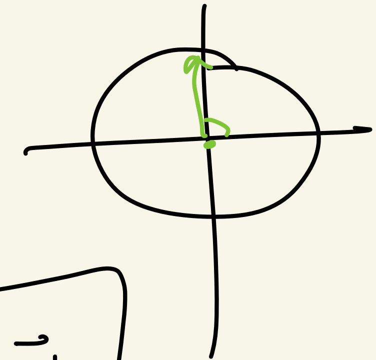
$$\theta = \cos^{-1}\left(\frac{\langle \vec{v},\vec{w}\rangle}{\|v\| \, \|w\|}\right).$$

Cauchy-Schwartz inequality

Thm $|\langle \vec{v},\vec{w}\rangle| \leq \|v\| \, \|w\|$ and this is an equality only when v is parallel to w.

$(v = cw)$

Since $\cos\theta = \dfrac{\langle v,w\rangle}{\|v\|\,\|w\|}$, when are two vectors perpendicular? When is $\theta = \pi/2 = 90^\circ$?



$$\cos(\pi/2) = 0 = \frac{\langle v,w\rangle}{\|v\|\,\|w\|}$$

$\Longrightarrow$ $\boxed{\langle v,w\rangle = 0 \text{ means that } \vec{v} \perp \vec{w}.}$

Note: This depends on inner product!

If $\langle v,w\rangle = \vec{v}\cdot\vec{w}$ on $\mathbb{R}^2$

$\vec{v} = (1,0)$ $(1,0)\perp(0,1)$ because

$\vec{w} = (0,1)$ $(1,0)\cdot(0,1) = 0\cdot 1 + 1\cdot 0 = 0$

$$\langle v, w\rangle = v_1w_1 - v_1w_2 - v_2w_1 + 2v_2w_2 \qquad (3.1.1)$$

is an inner product

But $(1,0)$ and $(0,1)$ are not perpendicular in this inner product!

$$\langle (1,0), (0,1)\rangle = 1\cdot 0 - 1\cdot 1 - 0\cdot 0 + 2\cdot 0\cdot 1 = -1 \neq 0$$

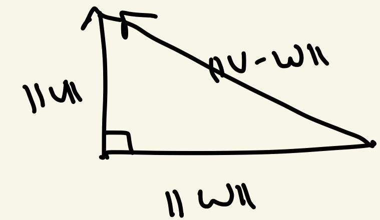
So perpendicularity depends on the inner product you've chosen!

Def We say $\vec{v}, \vec{w}$ are orthogonal if $\langle v, w\rangle = 0$.

(Perpendicular refers to dot product in particular.)

Thm Pythagorean Thm. Let V be a vector space w/ inner product $\langle -,- \rangle$. If v, w are orthogonal then

$$\|v-w\|^2 = \|v\|^2 + \|w\|^2$$



by orthogonality

Pf

$$\|v-w\|^2 = \langle \vec{v}-\vec{w}, \vec{v}-\vec{w} \rangle = \langle v,v \rangle - 2\overset{0}{\cancel{\langle v,w \rangle}} + \langle w,w \rangle$$

(we saw this 10/14)

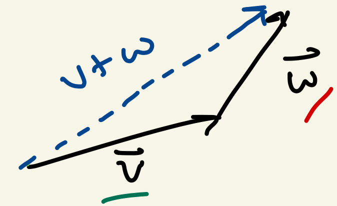
$$= \|v\|^2 + 0 + \|w\|^2$$

$$= \|v\|^2 + \|w\|^2$$

□

Thm Triangle Inequality, given any $v, w \in V$ and any $\langle -, - \rangle$.

$$\|v+w\| \leq \|v\| + \|w\|$$



Pf

$$(\|v+w\|)^2 = \langle v+w, v+w \rangle = \langle v,v \rangle + 2\langle v,w \rangle + \langle w,w \rangle$$

$$= \|v\|^2 + 2\langle v,w \rangle + \|w\|^2$$

Apply C-S

$|\langle v,w \rangle| \leq \|v\| \|w\|$

$$\leq \|v\|^2 + 2\|v\| \|w\| + \|w\|^2$$

$$= (\|v\| + \|w\|)^2$$

Take the sq root of both sides

$$\|v+w\| \leq \|v\| + \|w\|$$

$\square$

The dot product is just one example of an inner product.

$V = \mathbb{R}^2$

✓ $v \cdot w = v_1 w_1 + v_2 w_2$

✓ $\langle v, w \rangle = 5 v_1 w_1 + 2 v_2 w_2$ (weighted dot product)

✓ $\langle v, w \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 2 v_2 w_2$ (3.1.1)

✗ $\langle v, w \rangle = (v_1^2 + w_1^2)(v_2^2 + w_2^2)$ not bilinear

✗ $\langle v, w \rangle = -v_1 w_1 - v_2 w_2$ not positive

<u>Def</u> Define an <u>inner product space</u> as a vector space V with choice of inner product

Ex $V, \langle -,- \rangle = \mathbb{R}^2$, dot product

$V, \langle -,- \rangle = \mathbb{R}^2$, weighted dot product
$3v_1w_1 + 6v_2w_2$

Even though between these two examples, the vector space is the same, they are <u>different</u> <u>inner product spaces</u> because they have different inner products.

$$= C^0[a,b), \quad \langle f,g \rangle = \int_a^b f(x)g(x)\,dx$$

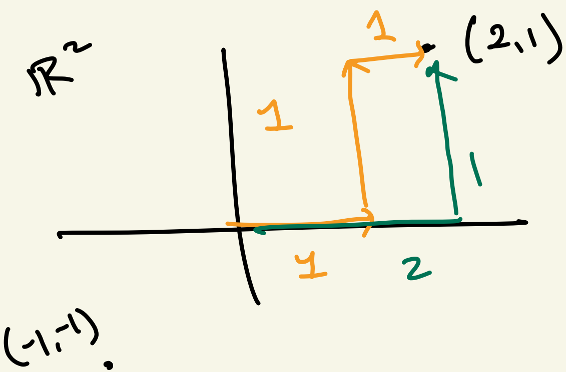
$$\neq C^0[a,b], \quad \langle f,g \rangle = \int_a^b f(x)g(x)e^{-x}\,dx$$

these are different inner product spaces

Norms

$\|v\| = \sqrt{\langle v, v\rangle}$ so far = distance from $\vec{0}$ to $\vec{v}$.

There's a notion of distance



$\|(2,1)\|_1$ = distance from (0,0) to (2,1) if you could only walk there on a grid

$\|(2,1)\|_1 = |2| + |1| = 3$

$\|(-1,-1)\|_1 = |-1| + |-1| = 2$

In general, the L^1 norm on $\mathbb{R}^n$ has formula

$$\|\vec{v}\|_1 = \sum_{i=1}^{n} |v_i| .$$

One can prove that

Positivity $\|v\|_1 > 0$ if $v \neq 0$. $\|\vec{0}\|_1 = 0$

Homog.

Δ ineq. $\|v + w\|_1 \leq \|v\|_1 + \|w\|_1$, So this $\|v\|_1 = \sum |v_i|$ is a coherent notion of distance

Is there an inner product $\langle -, - \rangle_1$ such that

$$\sum |v_i| = \|v\|_1 = \sqrt{\langle v, v \rangle_1}$$ ✗

$\langle -, - \rangle_1$ doesn't exist!

Some notions of distance don't have a corresponding notion of angle.

Some norms don't come from inner products.

<u>Define</u>: A normed vector space is a vector space w/ choice of norm, and a norm is a way to measure the magnitude of a vector in the following sense. $\|-\|$ norm

1) Positivity $\|v\| > 0$ if $v \neq 0$, $\|\vec{0}\| = 0$.

2) Homogeneity $\|c\vec{v}\| = |c| \, \|\vec{v}\|$, $c \in \mathbb{R}$ scalar

3) Δ inequality $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$.

Norm = just distance
no angle

Inner product = distance
angle

All inner products lead to norms

$$\langle -,- \rangle \implies \|-\| = \sqrt{\langle -,- \rangle}$$

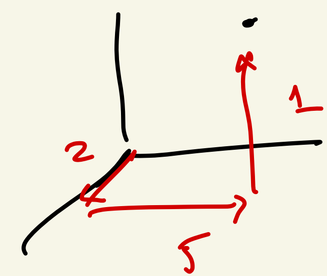
But not all norms lead to inner products.

$\sum |v_i| = \|v\|$ has no inner product.

(§3.3)

A, 75, 73 B, C

$$\|(2,5,1)\|_1 = |2| + |5| + |1| = 8$$



$$\|(2.5, 1)\|_1 = |2.5| + |1| = 3.5$$

$$\langle v, w \rangle = v_1 w_2 + v_2 w_1$$

$$v = (v_1 v_2)$$
$$w = (w_1 w_2)$$
$$u = (u_1 u_2)$$

$$\langle \overbrace{cv + dw}, u \rangle$$

$$= \langle (\underbrace{cv_1 + dw_1}, \underbrace{cv_2 + dw_2}), (\underbrace{u_1}; \underbrace{u_2}) \rangle$$

$$= (cv_1 + dw_1) u_2 + (cv_2 + dw_2) u_1$$

$$=$$

$$\langle \overbrace{cv + dw}, u \rangle$$

$$= \langle (\underbrace{cv_1 + dw_1}, \underbrace{cv_2 + dw_2}), (\underbrace{u_1}, \underbrace{u_2}) \rangle$$

$$= (cv_1 + dw_1)^2 u_1^2 + (cv_2 + dw_2)^2 u_2^2$$

$$\langle v, w \rangle = v_1^2 w_1^2 + v_2^2 w_2^2$$

$$\langle (v_1, v_2), (w_1, w_2) \rangle$$

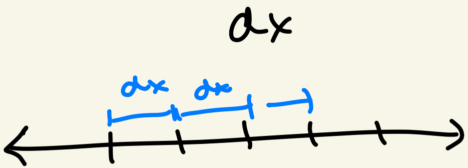
$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

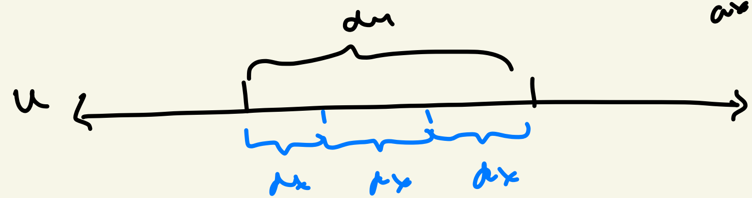
$$\langle \overbrace{cf + dg}, h \rangle$$

$$= \int_a^b (cf(x) + dg(x)) h(x) \, dx$$

$$\int f(x)\,dx \longleftrightarrow \int g(u)\,du \qquad du = 3\,dx$$

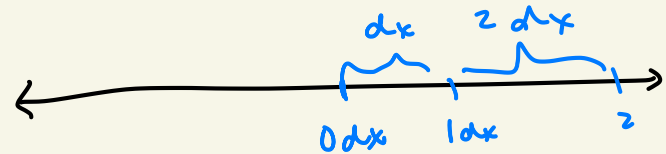
3 times stretched out version on x axis





$$\int f(x)\,3\,dx \qquad \int f(x)\,x\,dx$$

xdx



$$\langle f, g\rangle = \int f(x)g(x)\,\underbrace{e^{-x}\,dx}$$

$$\underline{\langle cf + dg, h\rangle} = \int \big(cf(x) + dg(x)\big)\,h(x)\,e^{-x}\,dx$$

$$= \int c\, f(x)h(x)e^{-x}\,dx + \int d\, g(x)h(x)e^{-x}\,dx$$

$$= c\int f(x)h(x)e^{-x}\,dx + d\int g(x)h(x)e^{-x}\,dx$$

$$= c\langle f,h\rangle + d\langle g,h\rangle$$

(a)

$$\langle f,f\rangle = \int_{-1}^{1} f(x)^2 e^{-x}\,dx \overset{?}{>} 0$$

(b)

$$\int_{-1}^{1} f(x)^2\, x\,dx \overset{?}{>} 0 .$$