Recall Normed vector spaces

- L^2 norm on $\mathbb{R}^n$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- L^1 norm on $\mathbb{R}^n$

$$\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

- L^∞ norm on $\mathbb{R}^n$

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}$$

E.g. $\|(-2, 1, -5, 3)\|_\infty = \max\{|-2|, |1|, |-5|, |3|\}$

$$= 5$$

More of this ...

consider $C^0[a,b]$ = continuous functions on $[a,b]$

- L^1 norm on $C^0[a,b]$

$$\|f\|_1 = \int_a^b |f(x)|\, dx$$

(Replace Σ with $\int$)

- L^2 norm on $C^0[a,b]$

$$\|f\|_2 = \sqrt{\int_a^b f(x)^2\, dx}$$

These are more common actually.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots \longrightarrow 0$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{1}{nx} \stackrel{?}{=} 0$$

- L^∞ norm on $C^0[a,b]$

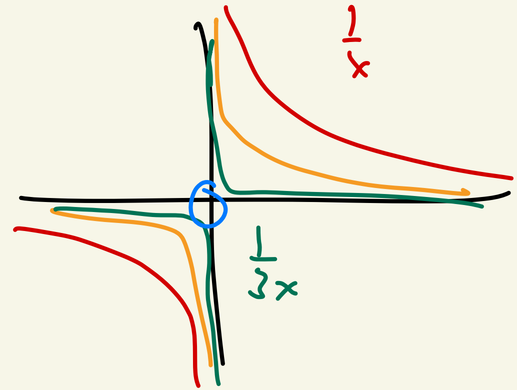
$$\|f\|_\infty = \max\{|f(x)|\}_{a \le x \le b}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} \frac{1}{nx} \stackrel{?}{=} 0$$

at $x = 0$

$$\frac{1}{nx} \not\to 0$$



$$\lim_{n\to\infty} \frac{1}{nx} = 0 \qquad \mathbb{R} / \{0\}$$

L^1, L^2, L^∞ are for situations like these

More generally, L^p norm

$$\|f\|_p = \sqrt[p]{\int_a^b |f(x)|^p \, dx} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

plug in $p=1$, L^1 norm, $p=2$, L^2 norm, $p=\infty$??

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$$

on $\mathbb{R}^n$

$$\|v\|_p = \sqrt[p]{|v_1|^p + |v_2|^p + \cdots + |v_n|^p}$$

if $|v_1|$ is the max, then $|v_1|^p$ will be much bigger than all the other $|v_i|$ as p gets bigger

$$2^{1000} \gg 1.99^{1000}$$

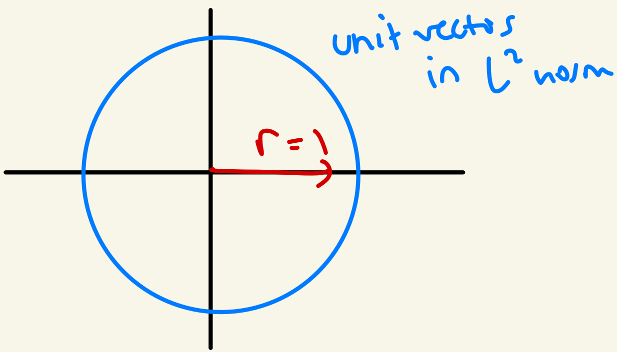
$$\|v\|_p \approx \sqrt[p]{|v_1|^p} = |v_1| = \max\{|v_1|, |v_2| \ldots |v_n|\}$$

$$= \|v\|_\infty$$

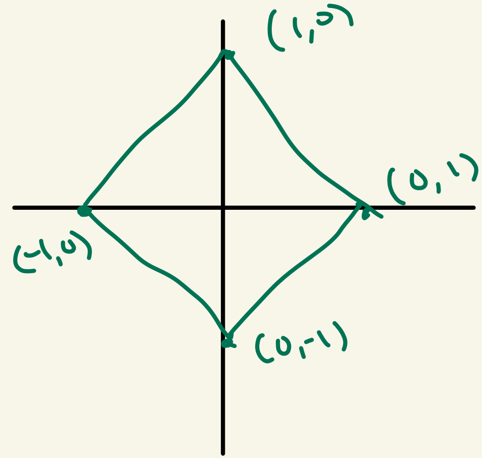
Unit vectors

The notion of a unit vector depends on what norm you've picked.

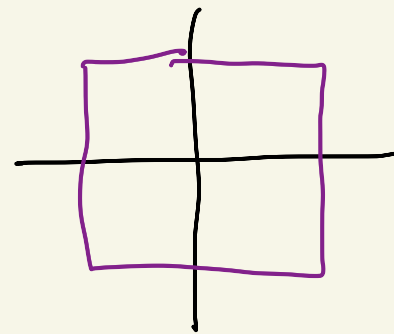
A vector $\vec{u}$ is a **unit vector** when $\|\vec{u}\|_* = 1$.



$\mathbb{R}^2, L^2$



$\mathbb{R}^2, L^1$



L^∞ norm

$$1 = \|(x,y)\|_1 = \boxed{|x| + |y| = 1}$$

3.4 Positive Definite Matrices

Bilinearity

$$\langle w, c_1 v_1 + c_2 v_2 \rangle = c_1 \langle w, v_1 \rangle + c_2 \langle w, v_2 \rangle$$

$c_1 = c_2 = 1$

$\langle w, v_1 + v_2 \rangle$

$= \langle w, v_1 \rangle + \langle w, v_2 \rangle$

$c_1 = 0$

$v_1 = 0$

$\langle w, c_2 v_2 \rangle = c_2 \langle w, v_2 \rangle$

Claim

$$\langle w, (c_1 v_1 + c_2 v_2) + c_3 v_3 \rangle$$

$$= \overbrace{\langle w, c_1 v_1 + c_2 v_2 \rangle} + \underline{\langle w, c_3 v_3 \rangle}$$

$$= c_1 \langle w, v_1 \rangle + c_2 \langle w, v_2 \rangle + c_3 \langle w, v_3 \rangle$$

This works for any linear combination!

<u>Claim</u> $$\left\langle \sum_{i=1}^{n} c_i \vec{v}_i , \sum_{j=1}^{m} d_j \vec{w}_j \right\rangle$$

you can FOIL

$$= \sum_{i,j=1}^{n,m} c_i d_j \langle \vec{v}_i , \vec{w}_j \rangle$$

sum over all combinations of terms.

general version of bilinearity

We can use this to our advantage!

Let $\langle -,- \rangle$ be an inner product on $\mathbb{R}^n$.

✓ $3x^2+4y^2$

✓ $3x^2-2xy+4y^2$

$\underline{x^2y+xy^2}$ ✗

Consider the standard basis on $\mathbb{R}^n$ $e_1, e_2, e_3, \dots, e_n$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$$

(unique linear combination of e_i which make x)

$$\underline{\langle \vec{x}, \vec{y} \rangle} = \langle x_1\vec{e}_1 + \dots + x_n\vec{e}_n, y_1\vec{e}_1 + \dots + y_n\vec{e}_n \rangle$$

$$= \left\langle \sum_{i=1}^{n} x_i \vec{e}_i \, , \, \sum_{j=1}^{n} y_j \vec{e}_j \right\rangle$$

Bilinearity

$$= \sum_{\substack{i=1 \\ j=1}}^{n,n} x_i y_j \langle \vec{e}_i , \vec{e}_j \rangle$$

If we want to compute $\langle \vec{x}, \vec{y} \rangle$, all we need to know are the values of $\boxed{\langle \vec{e}_i , \vec{e}_j \rangle}$ over all i and j.

The values of $\langle e_i, e_j \rangle$ determine the values of the inner product on the other vectors, $\langle x, y \rangle$.

Call $k_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$

then $\langle \vec{x}, \vec{y} \rangle = \underline{\sum_{i,j=1}^{n,n} k_{ij} x_i y_j}$

quadratic in $x_i y_j$, no other terms

What $k_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$ determine an inner product?

$\underline{Ex}$ $\underline{n=2}$ Let $\langle -, - \rangle$ be any inner product.

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\langle \vec{x}, \vec{y} \rangle = \langle x_1 \vec{e}_1 + x_2 \vec{e}_2 , y_1 \vec{e}_1 + y_2 e_2 \rangle$

$= x_1 y_1 \langle \vec{e}_1, \vec{e}_1 \rangle + x_2 y_1 \langle \vec{e}_2, \vec{e}_1 \rangle + x_1 y_2 \langle \vec{e}_1, \vec{e}_2 \rangle + x_2 y_2 \langle \vec{e}_2, \vec{e}_2 \rangle$

Claim $\langle \vec{e}_1, \vec{e}_1 \rangle$, $\langle \vec{e}_2, \vec{e}_1 \rangle$, $\langle e_2, \vec{e}_2 \rangle$ determine all other $\langle x, y \rangle$ values

Call $k_{11} = \langle e_1, e_1 \rangle$, $k_{12} = \langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = k_{21}$

$k_{22} = \langle e_2, e_2 \rangle$

$$\langle \vec{x}, \vec{y} \rangle = k_{11} x_1 y_1 + k_{12} x_1 y_2 + k_{12} x_2 y_1 + k_{22} x_2 y_2$$

These constants determine the inner product.

$$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\langle x, y \rangle = \vec{x}^T K \vec{y} \quad \text{where} \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle \end{pmatrix}$$

Summary: All inner products on $\mathbb{R}^n$ have the form

$$\langle \vec{x}, \vec{y} \rangle = x^T K y$$

where $K = \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \cdots & \langle e_1, e_n \rangle \\ \langle e_2, e_1 \rangle & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \langle e_n, e_1 \rangle & \cdots & \cdots & \langle e_n, e_n \rangle \end{pmatrix}$

Note that since inner products are symmetric, K is symmetric! $K^T = K$.

Positivity says that $\vec{x}^T K \vec{x} = \langle \vec{x}, \vec{x} \rangle > 0$ for $\vec{x} \neq 0$

If we know all inner products look like

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T K \vec{y}$. K symmetric

Which symmetric matrices make the formula $x^T K y$ into an inner product?

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(x_1 \quad x_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= x_1 y_1 + x_2 y_2 = x \cdot y \checkmark$$

Inner product!

$$K = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(x_1 \quad x_2)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= -x_1 y_1 - x_2 y_2 \quad \times$$

not positive!

Def We say a matrix K is positive definite if it is symmetric and $x^T K x > 0$ for all $\vec{x} \neq 0$.

Idea: Positive Definite matrices are exactly the symmetric matrices that arise from inner products $\langle \vec{x}, \vec{y} \rangle = x^T K y$.

$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is positive definite!

$K = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ not positive definite.