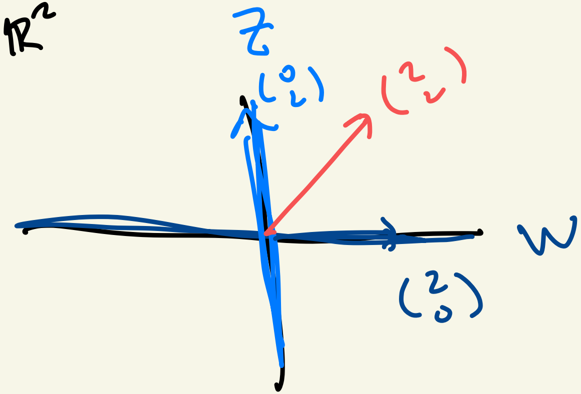
Warning: 2.2.22c (hard problem), probably won't pick this one to grade!

$Z \cap W$, $Z + W$ subspaces

$Z \cup W$ not a subspace

$\mathbb{R}^2$



$Z \cap W$ = x axis $\cap$ y axis = $\{(0,0)\}$ subspace (trivial subspace)

$Z + W = \mathbb{R}^2 = \binom{x}{y} = x\underbrace{\binom{1}{0}}_{\in W} + y\underbrace{\binom{0}{1}}_{\in Z}$

$\binom{2}{0} + \binom{0}{2} = \binom{2}{2} \notin Z \cup W.$

Prop ② is broken

But $Z \cup W$ = just axes and nothing else

Exam 1 : 10/9

① Required to have camera on that day

② Do in-Class exam (10:05 exam due time)

50 in-class

+ 10 upload time

Post study guide / practice exams later today.

5/6 problems (3/4 computation 2/3 short proof) → conceptual question

2.2.22 a/b length at most

<u>Last time:</u> All bases of a vector space have the same size!

If $\{v_1 \dots v_n\}$ and $\{w_1 \dots w_m\}$ are both bases of V abstract

then $n = m$. Kind of extraordinary

Basis size is inherent to V!

<u>Def:</u> The <u>dimension</u> of V is the size of any basis of V. <u>$\dim(V) = n$</u>.

<u>Pf</u> (Outline) ① Let $V = \mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\}$. row reduction matrices ...

Suppose $\{v_1 \dots v_k\}$ is a basis. Claim: $n = k$. <u>$\dim(\mathbb{R}^n) = n$</u>

All bases of $\mathbb{R}^n$ have size n.

$\vec{e}_1, \dots, \vec{e}_n$ e.g.

By def^n $\{v_1, \dots, v_k\}$ are independent and $\text{span}(v_1 \dots v_k) = \mathbb{R}^n$.

Independent $\Rightarrow$ $\left(c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0 \Rightarrow c_1 = \dots = 0 \right)$

$$\Rightarrow \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \vec{0} \in \mathbb{R}^n$$

has only trivial solution

$A = (v_1 \dots v_k)$ $\qquad A\vec{c} = \vec{0}$

$\Rightarrow A \longrightarrow \text{RREF}$ has no free variables!

$k < n$
$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$k = n$
$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
square

$k > n$
$$\begin{pmatrix} 1 & & & & \overbrace{* \quad *}^{\text{free columns}} \\ & 1 & & & * \quad * \\ & & 1 & & * \quad * \\ & & & 1 & \end{pmatrix}$$

RREF w/ no free variables $\Rightarrow$ all columns have leading 1's. $\Longrightarrow$ $k > n$ is false

$\Longrightarrow$ $k \leq n$.

$\text{Span}(v_1 \ldots v_k) = \mathbb{R}^n \Longrightarrow$ for all vectors $\vec{b}$ $\underline{\begin{pmatrix} v_1 \ldots v_k \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \vec{b}}$ always has at least 1 solution

Suppose $k < n$. Then $A = (v_1 \ldots v_k) \longrightarrow$ RREF have a row of 0's as the last row.

$k < n$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$k = n$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

square

$k > n$

$$\begin{pmatrix} 1 & & & & * & * \\ & 1 & & & * & * \\ & & 1 & & & \\ & & & 1 & * & * \end{pmatrix}$$

free columns

A, has a 0 row of zeros in RREF.

In particular

$$\left(\begin{array}{cccc|c} 1 & * & * & & 0 \\ & 1 & * & & 0 \\ & & 1 & & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

has no solution!

un row reducing

$$(\underline{A} \mid \vec{b}\,')$$

has no solution.

In conclusion RREF should have no row of 0's.

But the columns of A span

So it should have a solution!

$k < n$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$k = n$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

square

$k > n$

free columns

$$\begin{pmatrix} 1 & & & & * & * \\ & 1 & & & * & * \\ & & 1 & & * & * \\ & & & 1 & * & * \end{pmatrix}$$

So all bases of $\mathbb{R}^n$ have size n!

Ex $\underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\text{can't span even!}}$ can never be a basis of $\mathbb{R}^4$

only 3 vectors! I need at least 4 vectors to span $\mathbb{R}^4$.

Let $V = \mathbb{R}^n$. Consider $\{v_1 \ldots v_k\}$.

1) if $k > n$, then $v_1 \ldots v_k$ are dependent.

2) If $k < n$, then $\text{span}(v_1 \ldots v_k) \neq \mathbb{R}^n$, not a spanning set

3) All bases of $\mathbb{R}^n$ has size n.

Let V be arbitrary w/ two bases $\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_m\}$.

why is $m = n$?

V is arbitrary

row reduction

matrices

"Pretend"

$$v_1 \longrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

$$v_2 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$$

$$\vdots$$

$$v_n \longrightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

"linear transformation"

You can pretend V is just $\mathbb{R}^n$ in some coherent way.

$$\{w_1, \dots, w_m\} \longrightarrow \{\vec{w}_1, \dots, \vec{w}_m\} \in \mathbb{R}^n$$

basis $\Longrightarrow m = n$.

$\square$

Subspaces are vector spaces in and of themselves.

Given $W \subseteq \mathbb{R}^n$, $\dim(W)$ is well defined

$= \#$ of vectors in a basis of W.

Ex let $W = \text{span}\left(\begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\3\\4\end{pmatrix}\right)$. Find $\dim(W)$.

$\dim(W) = \#$ basis elements of W

$= \#$ independent vectors out of $\begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\3\\4\end{pmatrix}$

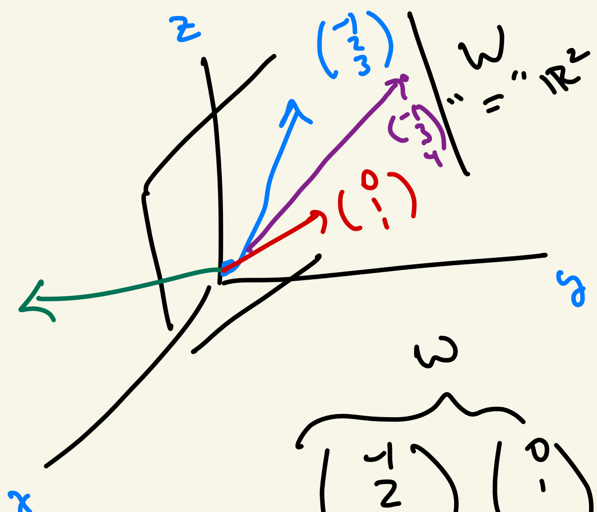
$$\begin{pmatrix}-1 & 0 & -1\\ 2 & 1 & 3\\ 3 & 1 & 4\end{pmatrix} \xrightarrow{\text{RREF}} \begin{array}{ccc} * & * & \text{free} \\ \end{array} \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \Longrightarrow \underline{\dim(W) = 2.}$$

$\neq R^3$

$\begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ is a basis for W.

$$\begin{pmatrix}-1\\3\\4\end{pmatrix} = 1\begin{pmatrix}-1\\2\\3\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$\begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$ is not a basis of $\mathbb{R}^3$, it's a basis of a smaller vector space inside of $\mathbb{R}^3$.



It's a basis of this plane that they form.

W is a slanted weird version of $\mathbb{R}^2$.

$$\overbrace{\begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}}^{W},$$

anything not in W will make it a basis of $\mathbb{R}^3$ ("adding 3rd dimension")

$$\underbrace{\begin{pmatrix}-1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}}_{\mathbb{R}^3}, \text{ eg}$$

Fundamental <u>Subspaces</u> of a matrix.

Let A be an $m \times n$ matrix. This is actually a function.

$$\begin{array}{ccc} \vec{x} & \longmapsto & A\vec{x} \\ T : \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

domain, set of all inputs

codomain, set where outputs live

$$\underset{n\times 1}{T(\vec{x})} = \underset{m\times \cancel{n}\ \ \cancel{n}\times 1}{A\vec{x}} \in \mathbb{R}^m$$

$$m \times 1$$

Every matrix is a function on vectors.

We can what we know about functions to A.

Next time ...

2.1.2

③ $\underbrace{\text{There exist a vector}}$ $\underline{\vec{0}}$ such that $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$.

$\vec{0}$ does not need to be $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. $f(\vec{x}) = 0$.

$\vec{0} = (1,1)$ in this weird formula for addition.

Standard +

$(x,y) + (u,v)$
$= (x+u, y+v)$

$\underline{\vec{0} = (0,0)}$

$(x,y) + (0,0)$
$= (x+0, y+0)$
$= (x,y)$

Weird +

not usual addition

$(x,y) + (u,v) = (xu, yv)$

$\vec{0} = (1,1)$

$(x,y) + \underline{(1,1)} = (x1, y1)$
$= (x,y)$

$-v = (-x, -y)$ $\qquad$ $-(x,y) = ??$

$\vec{0}$ role to play. In this weird addition $\vec{0} = (1,1)$ plays that role.

$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$. ← role

Let $\{\underbrace{\bar{v}_1 \ldots \bar{v}_n}_{\text{basis} \Rightarrow \text{ind, span } V}, \underline{w}\}$.

$\begin{pmatrix} | & & | \\ v_1 & \ldots & v_n \\ | & & | \end{pmatrix}$ will never have row of 0's.

$w = c_1 v_1 + \ldots + c_n v_n$. by def.

$v_1 \dots v_k$ basis $W \subset \mathbb{R}^n$ $\qquad W = \text{Span}(v_1 \dots v_k)$

Is $w \in W$?

$$\begin{pmatrix} v_1 \dots v_k & w \end{pmatrix} \xrightarrow{\text{yes}} \begin{pmatrix} 1 & & & & \boxed{*} \\ & 1 & & & \boxed{*} \\ & & 1 & & \boxed{*} \\ & & & 1 & \boxed{*} \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

$w \to$ free

$$\begin{pmatrix} v_1 \dots v_k & w \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

w 1

not in span