Last time: All inner products on $\mathbb{R}^n$ have the form

$$\langle \vec{x}, \vec{y} \rangle = x^T K y \quad \text{where}$$

$$K = \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \cdots & \underline{\langle e_1, e_n \rangle} \\ \vdots & \ddots & & \\ & & \ddots & \\ \underline{\langle e_1, e_n \rangle} & & & \langle e_n, e_n \rangle \end{pmatrix}$$

in particular if $k_{ij} = \langle e_i, e_j \rangle$

then $\langle \vec{x}, \vec{y} \rangle = \sum_{i,j}^{n} k_{ij} x_i y_j$

all inner products on $\mathbb{R}^n$ look like this!

If all inner products have the form $\langle \vec{x}, \vec{y} \rangle = \boxed{x^T K y}$,

which symmetric matrices K yield an inner product?

If $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then corresponding inner product is

the dot product!

$\vec{x} = (x_1, x_2)$

$\vec{y} = (y_1, y_2)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \langle \vec{x}, \vec{y} \rangle = x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y$$

$$= (x_1, x_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= (x_1, x_2) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 y_1 + x_2 y_2 \checkmark$$

The $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yields an inner product!

Def We say K is $\boxed{\text{positive}}$ <u>definite</u> if K is symmetric

$\boxed{x^T K x > 0}$ for all $\vec{x} \neq 0$.

Often $q(\vec{x}) = x^T K x$ is called the quadratic form associated to K.

Claim: If K is positive definite, then

$\langle \vec{x}, \vec{y} \rangle = \boxed{x^T K y}$ is an inner product.

If K is not positive def, it's not an inner product.

- Bilinearity
- Symmetry
- $\boxed{\text{positivity}}$

$x^T K y$ satisfies these properties only when K is pos def.

1) $\langle c\vec{x} + d\vec{y}, \vec{z} \rangle$

$$= (cx + dy)^T K z$$

Bilinearity

$$= (cx^T + dy^T) K z$$

$$= \left(c x^T K + d y^T K \right) z$$

$$= c x^T K z + d y^T K z = c \langle \vec{x}, \vec{z} \rangle + d \langle \vec{y}, \vec{z} \rangle$$

2) Symmetry, K is a symmetric matrix, $K^T = K$

$$\langle x, y \rangle = x^T K y = x^T K^T y = (Kx)^T y$$

$$\underbrace{\langle y, x\rangle}_{\in\ \mathbb{R}} = y^T K x = (y^T K x)^T = (Kx)^T y^{TT} = (Kx)^T y$$

$y^T K x \to |x|$ matrix

$$(x^T K x)^T = x^T K x$$

So $\langle x, y\rangle = \langle y, x\rangle$

3) Positivity if K is a positive definite matrix, by def $x^T K x > 0$; for all $x \neq 0$.

same!

Positivity axiom:

$\langle \vec{x}, \vec{x}\rangle = x^T K x > 0$ for all $x \neq 0$

So K being pos def is exactly saying that $\langle x, y\rangle = x^T K y$ satisfies positivity axiom

$\square$

Ex Are $\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ positive definite?

$\Updownarrow$

Are $\langle x, y \rangle = x^T \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} y$ and $\langle x, y \rangle = x^T \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} y$ inner products?

$x^T K x > 0$?

$$q(x) = x^T K x = (x_1 \;\; x_2) \begin{pmatrix} \overset{x_1^2}{\boxed{4}} & \overset{x_1 x_2}{-2} \\ -2 & \underset{x_2^2}{\boxed{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1 \;\; x_2) \begin{pmatrix} 4x_1 - 2x_2 \\ -2x_1 + 3x_2 \end{pmatrix}$$

$$= 4x_1^2 - 4x_1 x_2 + 3x_2^2 \qquad > 0 \; ?$$

$$= \left(4x_1^2 - 4x_1 x_2 + x_2^2\right) + 2x_2^2$$

$$= (2x_1 - x_2)^2 + 2x_2^2$$

completing the square strategy.

> 0 if $x_1, x_2 \neq 0$!

only way this can be 0 is if

$$2x_1 - x_2 = 0$$
$$x_2 = 0$$

$$2x_1 = 0 \quad \Rightarrow\!\!\!\!\!/ \ (x_1, x_2) \neq 0$$

So $\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$ is positive definite!

$\langle x, y \rangle = x^T \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} y$ is an inner product!

$$= \underline{4x_1y_1 - 2x_2y_1 - 2x_1y_2 + 3x_2y_2}$$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is <u>not</u> positive definite.

$$q(x) = x^T \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x = x_1^2 + 4x_1x_2 + x_2^2 > 0$$

when ?
$(x_1, x_2) \neq (0,0)$

If $(x_1, x_2) = (-1, 1)$

$$\Rightarrow \quad q(-1,1) = 1^2 + 4(-1)(1) + 1^2 =$$

$$2 - 4 = -2 < 0$$

So $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is not <u>positive definite</u> and

$$\langle x, y \rangle = x^T \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} y = x_1y_1 + 2x_1y_2 + 2x_2y_1 + x_2y_2$$

is <u>not</u> an inner product.

Proposition A 2×2 symmetric matrix is positive definite if and only if $a > 0$ $\quad ac - b^2 > 0$

$\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \det K$

"alternate definition" where $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

$$\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \rightsquigarrow \begin{array}{l} 4 > 0 \\ \det = 4 \cdot 3 - (-2)^2 = 12 - 4 = 8 > 0 \end{array}$$

$\Longrightarrow$ positive def.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \longrightarrow \begin{array}{l} 1 > 0 \\ 1 \cdot 1 - 2 \cdot 2 = -3 < 0 \end{array}$$ not positive def.

Proof Friday ...

Gram matrices Let V be any vector space w/ an inner product $\langle -,- \rangle$. Let $\vec{v}_1 \dots \vec{v}_k \in V$.

Define the Gram matrix of $\vec{v}_1 \dots \vec{v}_k$ to be

$$K = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_k \rangle \\ \vdots & \ddots & \\ \langle v_1, v_k \rangle & \cdots & \langle v_k, v_k \rangle \end{pmatrix}$$

$k \times k$ matrix

Thm $\underbrace{\vec{v}_1 \dots \vec{v}_k}_{\text{abstract vectors}}$ are independent iff $\underbrace{K}_{\text{actual matrix}}$ is positive definite.

no row reduction necessarily

Proof also on Friday!

$$K = \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_k\rangle \\ \vdots & \ddots & \\ \langle v_1, v_k\rangle & \cdots & \langle v_k, v_k\rangle \end{pmatrix} \quad k \times k \text{ matrix}$$

Thm $\vec{v}_1, \dots, \vec{v}_k$ are independent iff K is positive definite.

Ex Show that $\cos(x)$, $\cos(2x)$, $\cos(3x)$ are independent functions on $C^0[0, 2\pi]$. There's no trig identity between these function.

Define $\langle f, g\rangle = \int_0^{2\pi} f(x)\, g(x)\, dx$ $\quad L^2$ inner product on C^0.

$$K = \begin{pmatrix} \langle \cos x, \cos x\rangle & \langle \cos x, \cos(2x)\rangle & \langle \cos(x), \cos(3x)\rangle \\ \langle \cos x, \cos 2x\rangle & \langle \cos 2x, \cos 2x\rangle & \langle \cos(2x), \cos(3x)\rangle \\ \langle \cos x, \cos 3x\rangle & \langle \cos 2x, \cos 3x\rangle & \langle \cos 3x, \cos 3x\rangle \end{pmatrix} \quad 3 \times 3 \text{ matrix?}$$

$$\langle \cos x, \cos x \rangle = \int_0^{2\pi} \cos(x)\cos(x)\,dx = \int_0^{2\pi} \cos^2(x)\,dx = \pi$$

$$\langle \cos 2x, \cos 2x \rangle = \int_0^{2\pi} \cos^2 2x \,dx = \pi = \langle \cos(3x), \cos(3x) \rangle$$

$$\langle \cos x, \cos 2x \rangle = \int_0^{2\pi} \cos(x)\cos(2x)\,dx = 0$$

$$\langle \cos x, \cos 3x \rangle = \int_0^{2\pi} \cos(x)\cos(3x)\,dx = 0$$

$$\langle \cos 2x, \cos 3x \rangle = \int_0^{2\pi} \cos(2x)\cos(3x)\,dx = 0$$

cos x, cos 2x, cos 3x are inde when

$$K = \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$$ is positive def.

K is in fact positive definite!

K is pos def since $q(x) = x^T K x$

$$(x_1\ x_2\ x_3)\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \pi x_1^2 + \pi x_2^2 + \pi x_3^2 > 0.$$

!!

$\Longrightarrow$ $\cos x$, $\cos 2x$, $\cos 3x$ are independent functions!

by the theorem.

Thm $K = \begin{pmatrix} k_{11} & k_{12} & \cdots \\ k_{21} & k_{22} & \\ \vdots & & \ddots & \\ & & & k_{nn} \end{pmatrix}$

Define K_i to be top left $i \times i$ submatrix of K

$$K_2 = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

then K is pos def iff all $\det(K_i) > 0$.

Pf LU decompisition

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

at a critical pt (x_0, y_0)

then it's a min when $H(f)$ is pos def.

Is this an inner product? $*$

$$\langle v, w \rangle_1 = \frac{1}{4}\left(\|v+w\|_1 - \|v-w\|_1 \right)$$

where $\|-\|_1$ is the L^1 norm.

$$\|v\|_1 = \sum |v_i| = |v_1| + \dots + |v_n|$$

Is there an inner product $\langle -,- \rangle_1$ such that $\|v\|_1 = \sqrt{\langle v, v \rangle}$?

(a) If $\langle -,- \rangle_1$ were to exist, it have to have

$cv + dw$

$v + w$ $\qquad$ cv