Yesterday

Proposition Let $K = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a 2×2 symmetric matrix.

Then K is positive definite iff $a > 0$ and $\det K > 0$.

Pf Let $q(x) = x^T K x$

$$= (x_1 \;\; x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (x_1 \;\; x_2) \begin{pmatrix} ax_1 + bx_2 \\ bx_1 + cx_2 \end{pmatrix} = x_1(ax_1 + bx_2) + x_2(bx_1 + cx_2)$$

$$= \underline{ax_1^2 + 2b\,x_1x_2 + cx_2^2}$$

associated quadratic form to K.

K is positive definite $\iff$ $q(x) = ax_1^2 + 2bx_1x_2 + cx_2^2 > 0$
for $(x_1, x_2) \neq (0,0)$.

(definition of pos def)

For what a, b, c is
$ax_1^2 + 2bx_1x_2 + cx_2^2 > 0$?

Since $(x_1, x_2) \neq (0,0)$ one of x_1 or $x_2 \neq 0$.

Let's assume that $x_2 \neq 0$. (If $x_1 \neq 0$ then reverse argument.)

$$\frac{ax_1^2 + 2bx_1x_2 + cx_2^2}{x_2^2} > \frac{0}{x_2^2} \iff a\frac{x_1^2}{x_2^2} + 2b\frac{x_1x_2}{x_2^2} + c\frac{x_2^2}{x_2^2} > 0$$

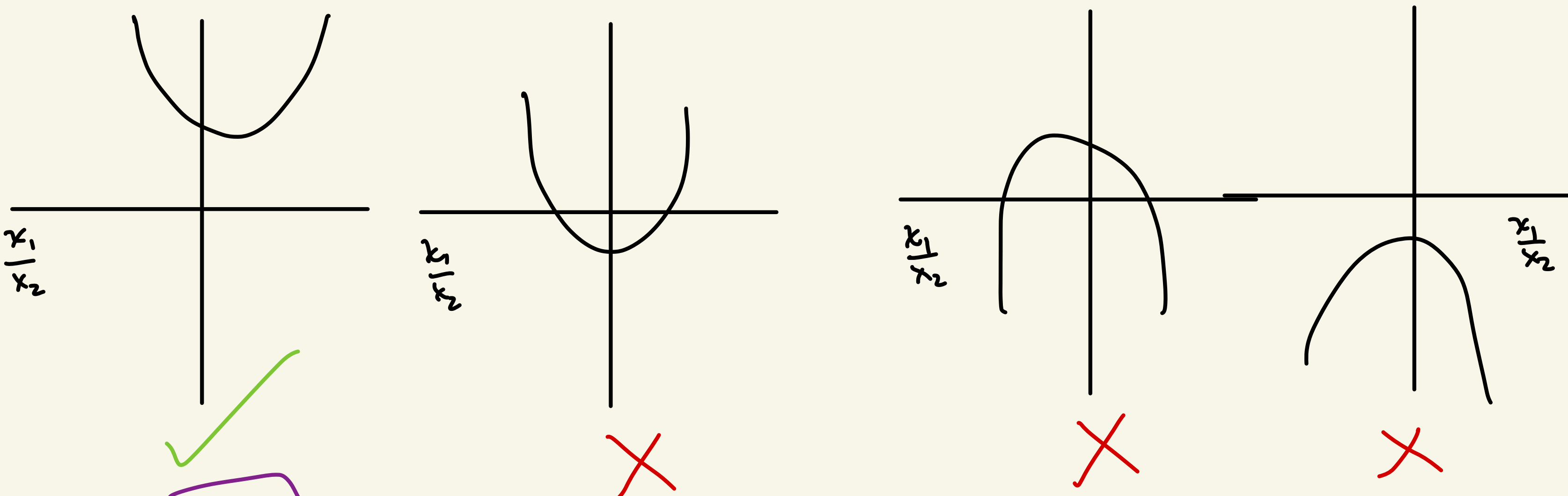
$$\iff a\left(\frac{x_1}{x_2}\right)^2 + 2b\left(\frac{x_1}{x_2}\right) + c > 0$$

$x_2^2 > 0$

This is a more typical polynomial in ratio $\frac{x_1}{x_2}$

$$a\left(\frac{x_1}{x_2}\right)^2 + 2b\left(\frac{x_1}{x_2}\right) + c > 0$$

parabola!



$\Longleftrightarrow$ $\boxed{a > 0}$ and parabola has no real roots

negative at some $\frac{x_1}{x_2}$ value.

$a\left(\frac{x_1}{x_2}\right)^2 + 2b\left(\frac{x_1}{x_2}\right) + c$ has no real roots

when

$$\frac{x_1}{x_2} = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$$ is imaginary. (QF)

which happens when $4b^2 - 4ac < 0$.

$$\Updownarrow$$

$$b^2 - ac < 0$$

$$\Updownarrow$$

$$\boxed{\det K = ac - b^2 > 0}$$

$\square$

Thm Let $v_1 \dots v_k$ be vectors in a vector space V w/ inner product $\langle -,- \rangle$. Then $v_1 \dots v_k$ are independent iff

the Gram matrix $K = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_k \rangle \\ \vdots & \ddots & \\ \langle v_1, v_k \rangle & & \langle v_k, v_k \rangle \end{bmatrix}$

is positive definite.

Last time Test independence $\cos(x), \cos(2x), \cos(3x)$

no row reduction?

Use this method!

Thm Let $v_1 \dots v_k$ be vectors in a vector space V w/ inner product $\langle -,- \rangle$. Then $v_1 \dots v_k$ are independent iff the Gram matrix $K = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_k \rangle \\ \vdots & \ddots & \\ \langle v_1, v_k \rangle & & \langle v_k, v_k \rangle \end{bmatrix}$ is positive definite.

$k \times k$

Pf Let $q(x) = x^T K x = (x_1 \dots x_k) \begin{pmatrix} \langle v_1 v_1 \rangle & \cdots & \vdots \\ \vdots & \ddots & \\ \vdots & \cdots & \langle v_k v_k \rangle \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}$

$$= (x_1 \dots x_k) \left(x_1 \begin{pmatrix} \langle v_1, v_1 \rangle \\ \langle v_1, v_2 \rangle \\ \vdots \\ \langle v_1, v_k \rangle \end{pmatrix} + x_2 \begin{pmatrix} \langle v_2 v_1 \rangle \\ \langle v_2 v_2 \rangle \\ \vdots \\ \langle v_2 v_k \rangle \end{pmatrix} + \dots + x_k \begin{pmatrix} \langle v_k v_1 \rangle \\ \vdots \\ \langle v_k v_k \rangle \end{pmatrix} \right)$$

$$= (x_1 \dots x_k) \left(\begin{pmatrix} \langle x_1 v_1 , v_1 \rangle \\ \vdots \\ \langle x_1 v_1 , v_k \rangle \end{pmatrix} + \dots + \begin{pmatrix} \langle x_k v_k , v_1 \rangle \\ \vdots \\ \langle x_k v_k , v_k \rangle \end{pmatrix} \right)$$

$$= (x_1 \dots x_k) \begin{pmatrix} \langle x_1 v_1 + x_2 v_2 + \dots x_k v_k, v_1 \rangle \\ \langle x_1 v_1 + x_2 v_2 + \dots + x_k v_k, v_2 \rangle \\ \vdots \\ \langle x_1 v_1 + \dots x_k v_k, v_k \rangle \end{pmatrix}$$

$$= (x_1 \dots x_k) \begin{pmatrix} \langle \sum_{i=1}^{k} x_i \vec{v}_i, \vec{v}_1 \rangle \\ \vdots \\ \langle \sum x_i \vec{v}_i, \vec{v}_k \rangle \end{pmatrix}$$

$$= x_1 \langle \sum x_i \vec{v}_i, v_1 \rangle + x_2 \langle \sum x_i \vec{v}_i, v_2 \rangle + \dots + x_k \langle \sum x_i \vec{v}_i, v_k \rangle$$

$$= \langle \underline{\sum x_i \vec{v}_i}, x_1 v_1 \rangle + \langle \underline{\sum x_i \vec{v}_i}, x_2 v_2 \rangle +$$

$$\dots + \langle \underline{\sum x_i \vec{v}_i}, x_k v_k \rangle$$

equal

$$= \langle \sum_{i=1}^{k} x_i \vec{v}_i, x_1 v_1 + x_2 v_2 + \dots + x_k v_k \rangle = \langle \sum_{i=1}^{k} x_i \vec{v}_i, \sum_{j=1}^{k} x_j \vec{v}_j \rangle$$

$$q(x) = x^T K x = \langle \sum_{i=1}^{k} x_i \vec{v}_i, \sum_{j=1}^{k} x_j \vec{v}_j \rangle$$

when positive?

$$K \text{ is pos def} \iff q(x) = x^T K x = \langle \sum_{i=1}^{k} x_i \vec{v}_i, \sum_{j=1}^{k} x_j \vec{v}_j \rangle > 0$$

$$\text{for } x \neq 0$$

1) If $v_1, \dots, v_k$ are independent, then the only way that

$\sum x_i \vec{v}_i = 0$ is if $x_1 = x_2 = \dots = x_k = 0$.

If $x \neq 0$ $\quad \sum x_i \vec{v}_i \neq 0 \Rightarrow q(x) = \|\sum x_i v_i\|^2 > 0$

since $\langle -,- \rangle$ was an inner product to begin with.

So K is positive definite!

2) If $v_1 \dots v_k$ were dependent. then there exists some

$(x_1, \dots x_k) \neq 0$ such that $\sum x_i v_i = 0$.

Call specific set of coefficients $\vec{b} = (b_1 \dots b_k) \neq 0$.

So $\sum b_i \vec{v}_i = 0$.

$$q(\vec{b}) = b^T K b = \left\langle \sum_{i=1}^{k} b_i \vec{v}_i, \sum_{j=1}^{k} b_j \vec{v}_j \right\rangle = \langle \vec{0}, \vec{0} \rangle = 0$$

Since $q(\vec{b}) = 0$ then $q(\vec{x}) \not> 0$ and

K is <u>not</u> positive definite. $\square$

3.6 Complex vector spaces

Typically $\mathbb{R}$ is the set of scalars.

But just as fine if $\mathbb{C}$ = complex numbers were the scalars.

$$M = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} \xrightarrow{ir_1 + r_2} \begin{bmatrix} 1 & i \\ 0 & -1 \end{bmatrix}$$

$$\xrightarrow{-1r_2} \begin{bmatrix} \boxed{1} & i \\ 0 & \boxed{1} \end{bmatrix}$$

2 leading 1's so $\operatorname{rk} M = 2$.

Everything in Chapters 1 and 2 is exactly the same if you replace $\mathbb{R}$ w/ $\mathbb{C}$.

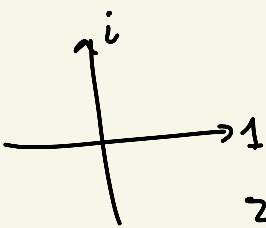
Two complications.

Exercise: 2.1.1 Show $\mathbb{C}$ are one real vector space.

$\vec{v} = (v_1 + i v_2)$

Turns out that $1, i$ form a basis of $\mathbb{C}$ as a real vector space.

Then $\dim_{\mathbb{R}} \mathbb{C} = 2$.



2D

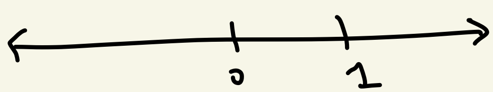
But what if we view $\mathbb{C}$ as a vector space w/ complex coefficients?

$(1+0i)$ and $(0+1i)$ are no longer independent!

$$i(1+0i) = i = (0+1i)$$

So all of a sudden $v = (1+0i)$ is a basis for $\mathbb{C}$ w/ complex coefficients.

$$\dim_{\mathbb{C}} \mathbb{C} = 1.$$



2) Second complication.

$\mathbb{R}^n$, dot product $\rightsquigarrow$ $\mathbb{C}^n$, dot product.

$\vec{z} \in \mathbb{C}^3$

$\vec{z} = (1+i,\ 2-i,\ 3+5i) \in \mathbb{C}^3$

$(1+i)(1+i)$

$1+2i-1 = 2i$

$(2-i)(2-i)$

$4-4i-1$

$9-25+30i$

wrong formula

$\times\ \vec{z}\cdot\vec{w} = z_1w_1 + \dots + z_nw_n$

$$\|z\| = \sqrt{(1+i)^2 + (2-i)^2 + (3+5i)^2}$$

$$= \sqrt{2i + 3-4i + -16+30i}$$

$$= \sqrt{-13+28i} \quad ???$$

Not a magnitude! Doesn't measure distance!

$\mathbb{C}^n \quad \vec{z}, \vec{w}$

necessary

$$\vec{z} \cdot \vec{w} = z_1 \overline{w_1} + \dots + z_n \overline{w_n}$$

$$\overline{(x+iy)} = x - iy$$

$$\vec{z} = (2-i, 1+i) \in \mathbb{C}^2$$

$$\|\vec{z}\| = \sqrt{z_1\overline{z_1} + z_2\overline{z_2}}$$

$$= \sqrt{(2-i)(2+i) + (1+i)(1-i)}$$

$$= \sqrt{5 + 2} = \sqrt{7}$$

measure of distance!

$C^0[a,b] \qquad e^{ix}, e^{i\phi x}$

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

instead! L^2 inner product $C^0_{\mathbb{C}}[a,b]$

1b Show that $\frac{1}{4}\left(\|v+w\|_1^2 - \|v-w\|_1^2 \right)$ is not an inner product.

Suppose $v = (v_1, v_2)$

$w = (w_1, w_2)$

$$\langle v, w\rangle_1 = \frac{1}{4}\left(\left(|v_1+w_1| + |v_2+w_2| \right)^2 - \left(|v_1-w_1| + |v_2-w_2| \right)^2 \right)$$

- not bilinear
- symmetric ✓
- positive ✓

$$\langle cv, w\rangle_1 \neq c\langle v, w\rangle_1$$

$$\langle (v, w \rangle_1 = \frac{1}{4}$$

$$\frac{1}{4}\left(\left(|cv_1+w_1| + |cv_2+w_2|\right)^2 - \left(|cv_1-w_1| + |cv_2-w_2|\right)^2\right)$$

$c = 7 \quad v = (1,0)$

$$a(b+c) = ab + ac$$

$$\underline{\langle v+w\ u\rangle} \overset{??}{=} \langle v,u\rangle + \langle w,u\rangle$$ Test this yourself?

$$\langle(\underline{v_1+w_1}, \underline{v_2+w_2}), (u_1, u_2)\rangle = 3(v_1+w_1)u_1 + 5(v_2+w_2)u_2$$

$$= 3v_1u_1 + 3w_1u_1 + 5v_2u_2 + 5w_2u_2$$

$$3 \times 5 \qquad \begin{matrix} | & | & | \\ | & | & | \\ | & | & | \\ | & | & | \\ | & | & | \end{matrix} \qquad = 15$$

$$a(b+c) \qquad ab + ac$$

$$\langle v, w+u \rangle = \langle v, w \rangle + \langle v, u \rangle$$

$$ba + ca$$

$$= \langle w, v \rangle + \langle u, v \rangle$$

$$(b+c)a$$

$$= \langle w+u, v \rangle$$

$$\langle v, w \rangle + \langle w, v \rangle = 2\langle v, w \rangle$$

$v_1 \, v_2 \, v_3$ $\qquad \langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$

$1, e^x, e^{2x}$

$$K = \begin{pmatrix} \langle 1,1 \rangle & \langle 1, e^x \rangle & \langle 1, e^{2x} \rangle \\ \langle 1, e^x \rangle & \langle e^x, e^x \rangle & \langle e^x, e^{2x} \rangle \\ \langle \; - & - \; - & - \end{pmatrix}$$

$$= \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Is this positive definite?

Apply thm $\quad \underline{K \text{ pos}} \iff \underline{1, e^x, e^{2x}}$ independent.

argue why these are 3 independent vectors.

$$\langle a1 + be^x + ce^{2x}, a + be^x + ce^{2x} \rangle = \| a + be^x + ce^{2x} \|^2 > 0$$

$$= (a\ b\ c)\, K \begin{pmatrix} a \\ b \\ c \end{pmatrix} > 0$$

$$(a, b, c) \neq \vec{0}$$

If $a + be^x + ce^{2x} = 0$ then $a, b, c = 0$.

as function