Last time : Chap 3 done! Complex vector spaces

↳ come back later

Chapter 4 §4.1 Orthogonal bases and orthonormal bases

Recall in an inner product space, v, w are orthogonal when $\langle v, w \rangle = 0$. orthogonal "=" perpendicular

general dot product

Def : A basis $\vec{v}_1 \dots \vec{v}_n$ is called an orthogonal basis if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for all $i \neq j$.

A basis $\vec{u}_1, \dots \vec{u}_n$ is called orthonormal if

it is orthogonal $\left(\langle \vec{u}_i, \vec{u}_j \rangle = 0 \right)$ and

$\|u_i\| = 1$. (Basis vectors are unit vectors.)

Ex Let $V = \mathbb{R}^n$ w/ dot product. Then $\vec{e}_1, \dots, \vec{e}_n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$ is an

Orthonormal basis.

In fact e_i are orthogonal to each other.

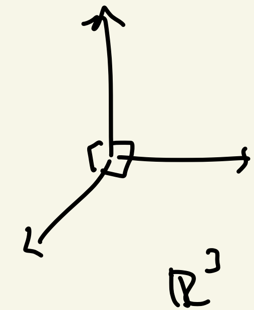
$$e_i \cdot e_j = e_i^T e_j = \begin{array}{c} (0 \; 0 \dots 1 \dots 0) \\ \scriptstyle i^{th} \end{array} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} j^{th} \qquad i \neq j$$

$$= 0\cdot 0 + \dots + \underset{i^{th}}{0\cdot 1} + \dots + \underset{j^{th}}{1\cdot 0} + \dots + 0\cdot 0$$

$$= 0 \qquad \text{So } e_i \perp e_j$$

$$\| e_i \| = \sqrt{0^2 + 0^2 + \dots + 1^2 \dots + 0^2}$$

$$= \sqrt{1^2} = 1.$$



Since $e_1 \dots e_n$ are orthogonal to each other and they're unit vectors, they form an orthonormal basis.

This is the motivating example.

Suppose $V = \mathbb{R}^2$ $\quad \langle (v_1 v_2), (w_1 w_2) \rangle = 2v_1w_1 + 3v_2w_2.$

Claim: $\vec{e}_1, \vec{e}_2$ is still orthogonal, but not orthonormal!

$$\langle e_1, e_2 \rangle = \langle (1,0), (0,1) \rangle$$
$$= 2 \cdot 1 \cdot 0 + 3 \cdot 0 \cdot 1 = 0 \qquad \text{still orthogonal.}$$

But $\|e_1\| = \sqrt{\langle e_1, e_1 \rangle} = \sqrt{2 \cdot 1 \cdot 1 + 3 \cdot 0 \cdot 0}$

$$= \sqrt{2} \neq 1$$

$$u_1 = \frac{e_1}{\|e_1\|} = \left(\frac{1}{\sqrt{2}}, 0\right) \qquad u_2 = \frac{e_2}{\|e_2\|} = \frac{(1,0)}{\sqrt{3}}$$
$$= \left(\frac{1}{\sqrt{3}}, 0\right)$$

So $\left(\frac{1}{\sqrt{2}}, 0\right), \left(0, \frac{1}{\sqrt{3}}\right)$ is an orthonormal basis of $\mathbb{R}^2$ w/r/t $2v_1w_1 + 3v_2w_2$.

Ex $V = \mathbb{R}^3$ w/ dot product

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

is an orthogonal basis.

automatically independent!

$$v_1 \cdot v_2 = 1 \cdot 0 + 2 \cdot 1 - 1 \cdot 2 = 0$$

$$v_1 \cdot v_3 = 1 \cdot 5 - 2 \cdot 2 - 1 \cdot 1 = 0$$

$$v_2 \cdot v_3 = 0 \cdot 5 - 2 \cdot 1 + 2 \cdot 1 = 0$$

They tend to look like this, lots of sqrts.

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad u_3 = \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

This is an orthonormal basis.

Def : We say that a set of vectors $v_1 \dots v_k$ are
<u>mutually orthogonal</u> if $\langle v_i, v_j \rangle = 0 \quad i \neq j$.

Note : Orthogonal basis is formed by a basis of mutually orthogonal vectors.

Prop Let $v_1 \dots v_k$ be mutually orthogonal. Then they are an independent set.

Pf : Suppose $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = 0$. WTS $c_i = 0$.

On the one hand $\langle \vec{v}_i, c_1 v_1 + \dots + c_k v_k \rangle$

$$= \langle \vec{v}_i, 0 \rangle = 0$$

OTOH

$$\langle v_i, c_1v_1 + \cdots + c_kv_k\rangle = c_1\langle v_i, v_1\rangle + \cdots + c_k\langle v_i, v_k\rangle$$

$$= c_1 \cdot 0 + \cdots + c_i\langle v_i, v_i\rangle + \cdots + c_k \cdot 0$$

since $\vec{v}_i$ are mutually orthogonal $(\langle v_i, v_j\rangle = 0)$.

$$= c_i \|\vec{v}_i\|^2$$

$$c_i\|v_i\|^2 = 0$$

Assuming $\vec{v}_i \neq 0$, then $\|v_i\|^2 \neq 0$ by positivity

$\Rightarrow$ $c_i = 0$. True for all i. $\square$

Prop Suppose $\dim V = n$, and $\vec{v}_1, \ldots, \vec{v}_n$ is a mutually orthogonal set of n vectors. Then this a orthogonal basis.

Pf: mutually orthogonal $\Longrightarrow$ independent

n ind. vectors
$\dim V = n$ $\Longrightarrow$ basis $\square$

Ex $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$, 3 mutually orthogonal vectors in $\mathbb{R}^3$

automatically form a basis!

Thm Suppose $\vec{u}_1, \dots, \vec{u}_n$ is an orthonormal basis of V.

Then $\forall\, v \in V$

$$v = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n$$

where $c_i = \langle \vec{v}, \vec{u}_i \rangle$.

Important!
no row reduction!

Furthermore, $\|v\| = \sqrt{c_1^2 + c_2^2 + \dots + c_n^2}$.

Despite the fact $\langle -,- \rangle$ was general, it still looks like our usual formula.

Pf Let $\vec{u}_1, \dots, \vec{u}_n$ be the orthonormal basis.

Since it's a basis

$$v = c_1\vec{u}_1 + \dots + c_n\vec{u}_n.$$

But since they're orthonormal,

$$\langle \vec{v}, \vec{u}_i \rangle = \langle c_1\vec{u}_1 + \dots + c_n\vec{u}_n, \vec{u}_i \rangle$$

$$= c_1 \cancel{\langle u_1, u_i \rangle}^{0} + \dots + c_i\langle u_i, u_i \rangle + \dots + c_n \cancel{\langle u_n, u_i \rangle}^{0}$$

$$= c_i \langle u_i, u_i \rangle = c_i \|u_i\|^2$$ (u_i is a unit vector)

$$= c_i \cdot 1^2 = c_i.$$

$$\|v\|^2 = \langle v, v\rangle = \left\langle \sum_{i=1}^{n} c_i u_i , \sum_{j=1}^{n} c_j u_j \right\rangle$$

$$= \sum_{i,j=1}^{n} c_i c_j \langle u_i , u_j\rangle$$

$\langle u_i, u_j\rangle = 0$ if $i \neq j$

$\langle u_i, u_j\rangle = 1$ $i = j$

$$= c_1^2 \cdot \underbrace{\langle u_1, u_1\rangle}_{1} + \dots + c_n^2 \cdot \underbrace{\langle u_n, u_n\rangle}_{1}$$ by orthonormal

$$= c_1^2 + c_2^2 + \dots + c_n^2$$

$$\Longrightarrow \quad \|v\| = \sqrt{c_1^2 + \dots + c_n^2} \, .$$

$\square$

Ex $\quad u_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} \quad u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} \quad u_3 = \frac{1}{\sqrt{30}}\begin{pmatrix}5\\-2\\1\end{pmatrix}.$

Suppose $v = (1,1,1)$. Write v as a linear combination of u_1, u_2, u_3.

Before...

$$\left(\begin{array}{cccc} \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} & 1 \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} & 1 \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & 1 \end{array}\right) \xrightarrow[\text{reduce}]{\text{row}} \text{:(}$$

Not necessary!

$$c_i = \langle \vec{v}, \vec{u}_i \rangle$$

$$u_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} \qquad u_3 = \frac{1}{\sqrt{30}}\begin{pmatrix}5\\-2\\1\end{pmatrix}. \qquad v = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$$c_1 = v\cdot u_1 = \frac{1}{\sqrt{6}}(1+2-1) = \frac{2}{\sqrt{6}}$$

$$c_2 = v\cdot u_2 = \frac{1}{\sqrt{5}}(0+1+2) = \frac{3}{\sqrt{5}}$$

$$c_3 = v\cdot u_3 = \frac{1}{\sqrt{30}}(5-2+1) = \frac{4}{\sqrt{30}}$$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{2}{\sqrt{6}}u_1 + \frac{3}{\sqrt{5}}u_2 + \frac{4}{\sqrt{30}}u_3$$

$$= \frac{2}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} + \frac{3}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{pmatrix}0\\1\\2\end{pmatrix} + \frac{4}{\sqrt{30}}\frac{1}{\sqrt{30}}\begin{pmatrix}5\\-2\\1\end{pmatrix}$$

$$6 \qquad\qquad 5 \qquad\qquad 30$$

$$\| (1,1,1) \| = \sqrt{\left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{3}{\sqrt{5}}\right)^2 + \left(\frac{4}{\sqrt{30}}\right)^2}$$

$$= \sqrt{\frac{4}{6} + \frac{9}{5} + \frac{16}{30}}$$

$$= \sqrt{\frac{20 + 54 + 16}{30}} = \sqrt{\frac{90}{30}} = \sqrt{3}$$

$$= \sqrt{1^2 + 1^2 + 1^2}$$

It works!

3.6.30b

$$\begin{pmatrix} 2 & -1+i & 1-2i \\ -4 & 3-i & 1+i \end{pmatrix}$$

$2(-1+i)+3-i$

$1+i$

$$\xrightarrow{2r_1+r_2} \begin{pmatrix} 2 & -1+i & 1-2i \\ 0 & 1+i & * \end{pmatrix}$$

leading 1

$$\frac{1}{1+i} r_2 = \frac{1-i}{2} r_2 \longrightarrow$$

$$\frac{1}{1+i} = \frac{1-i}{\underline{(1+i)(1-i)}} = \frac{1-i}{\underline{1^2+1^2}} = \frac{1-i}{2}$$

$$\frac{1}{x+iy} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$(x+iy)(x-iy) = x^2 + ixy - ixy - (iy)^2$$

$$= x^2 - -1y^2 = x^2 + y^2$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} a \\ b \\ 2 \end{pmatrix}, \begin{pmatrix} b \\ -2 \\ b \end{pmatrix}$$

what a, b make this orthogonal?

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 1 \\ 2 \end{pmatrix} = 0$$

$$a + 2 - 2 = 0$$

$$\Longrightarrow a = 0$$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} b \\ -2 \\ 1 \end{pmatrix} = b - 4 - 1 = 0$$

$$\Rightarrow b = 5.$$

$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \qquad\qquad a + 2b - c = 0$$

$$\Rightarrow \qquad\qquad 0a + b + 2c = 0$$

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

now reduce $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$ to solve!

1 free variable!

$$(a, b, c) = (1, 2, -1) \times (0, 1, 2)$$