If $u_1 \ldots u_n$ is an orthonormal basis of a vector space

V, $\quad v = c_1 u_1 + \ldots + c_n v_n$ where

$$c_i = \langle v, u_i \rangle$$

(no row reduction only $\langle -,- \rangle$!)

and

$$\|v\| = \sqrt{c_1^2 + c_2^2 + \ldots + c_n^2}.$$

(looks like $\mathbb{C}^2$ norm)

despite that $\langle -,- \rangle$ might not be the L^2 inner product!

$V = \mathcal{P}^{(2)}$ = degree 2 or less polynomials in 1 variable.

$= \{ a + bx + cx^2 \}$

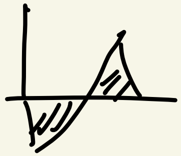
$= \text{span}(1, x, x^2) \subseteq C^0[0,1]$ Viewing polynomials as functions!

$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$

Claim $p_1 = 1 \qquad p_2 = 2x - 1 \qquad p_3 = 6x^2 - 6x + 1$

is an orthogonal basis of $\mathcal{P}^{(2)}$ w/r/t L^2-inner product!

$\langle p_1, p_2 \rangle = \int_0^1 1(2x-1)\, dx = 0$



$$\langle p_1, p_3 \rangle = \int_0^1 1(6x^2-6x+1)\,dx = 0$$

$$\langle p_2, p_3 \rangle = \int_0^1 (2x-1)(6x^2-6x+1)\,dx = 0$$

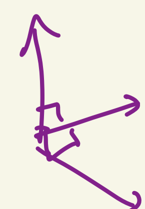
$1, x, x^2$ basis

3 mutually orthogonal vectors in a 3-dim vector space automatically form a orthogonal basis!

orthonormal basis of $P^{(2)}$ (version of e_1, e_2, e_3)

$$u_1 = \frac{p_1}{\|p_1\|} = \frac{1}{\sqrt{\int_0^1 1^2\,dx}} = 1$$

$$u_2 = \frac{p_2}{\|p_2\|} = \frac{2x-1}{\sqrt{\int_0^1 (2x-1)^2\,dx}} = \frac{1}{\sqrt{\frac{1}{3}}}(2x-1) = \sqrt{3}(2x-1)$$



$$u_3 = \frac{p_3}{\|p_3\|} = \frac{6x^2-6x+1}{\sqrt{\int_0^1 (6x^2-6x+1)\,dx}} = \sqrt{5}\,(6x^2-6x+1)$$

$v = 1 + x + x^2$ as a linear combination of $u_1 u_2 u_3$.

we know that $v = c_1 u_1 + c_2 u_2 + c_3 u_3$

$c_1 = \langle v, u_1 \rangle \quad c_2 = \langle v, u_2 \rangle \quad c_3 = \langle v, u_3 \rangle$

$c_1 = \int_0^1 (1+x+x^2)(1)\,dx = \frac{11}{6} \quad c_2 = \int_0^1 (1+x+x^2)(\sqrt{3}(2x-1))\,dx$

$= \frac{\sqrt{3}}{3}$

$c_3 = \int_0^1 (1+x+x^2)(\sqrt{5}(6x^2-6x+1))\,dx = \frac{\sqrt{5}}{30}$

$1+x+x^2 = \frac{11}{6}(1) + \frac{\sqrt{3}}{3}\left(\sqrt{3}(2x-1)\right) + \frac{\sqrt{5}}{30}\left(\sqrt{5}(6x^2-6x+1)\right)$

$c_1 u_1 + c_2 u_2 + c_3 u_3$

Claim

$$\| 1+x+x^2 \| = \sqrt{c_1^2 + c_2^2 + c_3^2}$$

$$\overset{\|}{\sqrt{\int_0^1 (1+x+x^2)^2 \, dx}} = \sqrt{\left(\frac{11}{6}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{5}}{30}\right)^2}$$

$$\overset{\|}{\sqrt{\frac{37}{10}}} = \sqrt{\frac{121}{36} + \frac{3}{9} + \frac{5}{900}} = \sqrt{\frac{37}{10}}$$

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§4.2 Gram-Schmidt Process

1) How do you make orthogonal / orthonormal bases in the first place?

2) What are they good for? partial answer (thm helpful)

Idea of Gram-Schmidt Process

Input : $w_1, \dots w_n$ basis of V.S $\langle -,- \rangle$

Output : $v_1 \dots v_n$ orthogonal basis

Recursive algorithm
- solve for v_1,
- solve for v_2 in terms of v_1
- solve for v_3 in terms of v_1, v_2

$\vdots$

- solve for v_n in terms $v_1, \dots, v_{n-1}$.

Formula!

Thm Gram-Schmidt

Given a basis $w_1, \dots, w_n$ of an inner product space

let $v_1 = w_1$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$ (in terms of v_1)

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$ (in terms of v_1, v_2)

$\vdots$

$$v_n = w_n - \sum_{i=1}^{n-1} \frac{\langle w_n, v_i \rangle}{\|v_i\|^2} v_i .$$ (in terms of $v_1, \dots, v_{n-1}$)

then $v_1, \dots, v_n$ is an orthogonal basis!

Pf Outline

Let $v_1 = w_1$.

Suppose $v_2 = w_2 - c v_1$.

(hoping for the best) maybe we can find a c that makes $v_1 \perp v_2$.

$$\langle v_1, v_2 \rangle = 0$$

$$\Rightarrow \langle v_1, w_2 - c v_1 \rangle = 0$$

$$\Rightarrow \langle v_1, w_2 \rangle - c \langle v_1, v_1 \rangle = 0$$

$$\Rightarrow \langle v_1, w_2 \rangle - c \|v_1\|^2 = 0$$

$$c = \frac{\langle v_1, w_2 \rangle}{\|v_1\|^2}$$

Therefore let

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

and $v_1 \perp v_2$ by construction!

Hope $v_3 = w_3 - c_1 v_1 - c_2 v_2$

$\langle v_1, v_3 \rangle = 0 \rightsquigarrow c_1 = \dfrac{\langle w_3, v_1 \rangle}{\|v_1\|^2}$

$\langle v_2, v_3 \rangle = 0 \rightsquigarrow c_2 = \dfrac{\langle w_3, v_2 \rangle}{\|v_2\|^2}$

So

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

and $v_1\ v_2\ v_3$ are mutually orthogonal by construction!

etc . . .

$\text{IOU}^{\square}$

$$v_1 = w_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\vdots$$

$$v_n = w_n - \sum_{i=1}^{n-1} \frac{\langle w_n, v_i \rangle}{\|v_i\|^2} v_i$$

<u>Ex</u> $\quad w_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad w_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

Output $v_1 v_2 v_3$ orthogonal basis

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

What's $\langle -,- \rangle$?

let's pick dot product for simplicity.

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$$

$$v_2 = \begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}}{\|(1,1,-1)\|^2}\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{-1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}3\\0\\6\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}4\\1\\5\end{pmatrix}$$

$$v_1 \cdot v_2 = (1,1,-1)\cdot\left(4/3, 1/3, 5/3\right) = \frac{4+1-5}{3} = 0\,!$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= \begin{pmatrix}2\\-2\\3\end{pmatrix} - \frac{(2,-2,3)\cdot(1,1,-1)}{\|(1,1,-1)\|^2}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \frac{\cancel{\tfrac{1}{3}}(2,-2,3)\cdot(4,1,5)}{\|\cancel{\tfrac{1}{3}}(4,1,5)\|^2}\cancel{\tfrac{1}{3}}\begin{pmatrix}4\\1\\5\end{pmatrix}$$

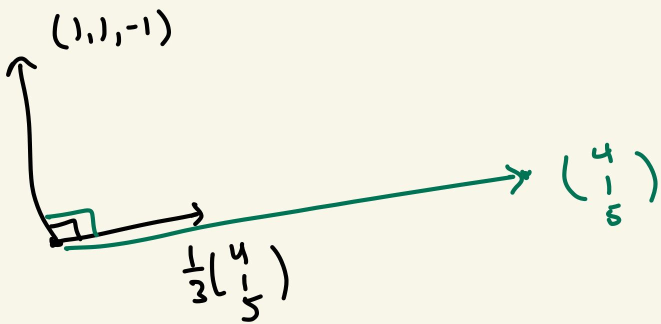
$$= \begin{pmatrix}2\\-2\\3\end{pmatrix} - \frac{-3}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \frac{21}{42}\begin{pmatrix}4\\1\\5\end{pmatrix} = \begin{pmatrix}2\\-2\\3\end{pmatrix} + \begin{pmatrix}1\\1\\-1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}4\\1\\5\end{pmatrix}$$

$$= \begin{pmatrix}3\\-1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}4\\1\\5\end{pmatrix} = \begin{pmatrix}3-2\\-1-1/2\\2-5/2\end{pmatrix} = \begin{pmatrix}1\\-3/2\\-1/2\end{pmatrix}$$

$$v_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix} \quad v_2 = \frac{1}{3}\begin{pmatrix}4\\1\\5\end{pmatrix} \quad v_3 = \begin{pmatrix}1\\-3/2\\-1/2\end{pmatrix}$$ is an orthogonal basis!

Note! After step 2 $v_2 = \frac{1}{3}\begin{pmatrix}4\\1\\5\end{pmatrix}$ Pretend $v_2 = \begin{pmatrix}4\\1\\5\end{pmatrix}$.

It'll still work!



Suppose $P^{(2)} = V$ $\quad \langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$

Start $1, x, x^2$ of $P^{(2)}$

$\int_0^1 1 \cdot x \, dx = \frac{1}{2} \neq 0$ $\quad 1 \not\perp x$. So apply G-S!

$$1, x, x^2 \xrightarrow{G\text{-}S} 1,\ 2x-1,\ 6x^2-6x+1.$$

natural basis, but it's not orthogonal!

We can use G-S to make one.

$L = AA^T$

$R = A^TA$.

G-S: The input $w_1, w_2 \ldots w_n$ needs to be a <u>basis!</u>

$v_1 = w_1$

$v_2 = c_1 w_1 + c_2 w_2$

$\vdots$

$v_n = c_1 w_1 + \ldots + c_n w_n$