Last time: Gram - Schmidt Process

Inputs: $w_1 \dots w_n$ a basis

Output: $v_1, \dots, v_n$ an orthogonal basis

$$v_1 = w_1$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

etc ..

Ex $P^{(2)} =$ degree ≤ 2 polynomials $\subseteq C^0[0,1]$

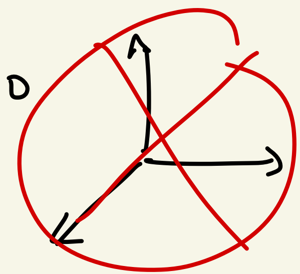
$$\begin{gathered} \rotatebox{90}{$=$} \\ \operatorname{span}(1, x, x^2) = \{ a1 + bx + cx^2 \} \end{gathered}$$

L^2 inner product $\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$

Natural basis of $P^{(2)}$ is $w_1 = 1$, $w_2 = x$, $w_3 = x^2$

Not orthogonal!

$$\langle w_1, w_2 \rangle = \langle 1, x \rangle = \int_0^1 1 \cdot x\, dx = \frac{1}{2} \neq 0$$



G-S will turn this into an orthog. basis!

$$v_1 = w_1 = \boxed{1}$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$\langle x, 1 \rangle$

$\langle 1, 1 \rangle$

$$= x - \frac{\int_0^1 x \cdot 1 \, dx}{\int_0^1 1^2 \, dx} 1 = x - \frac{1}{2} = \frac{1}{2}\boxed{(2x-1)}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$= x^2 - \frac{\int_0^1 x^2 \cdot 1 \, dx}{\int_0^1 1^2 \, dx} 1 - \frac{\int_0^1 x^2 (x - \frac{1}{2}) \, dx}{\int_0^1 (x-\frac{1}{2})^2 \, dx} (x - \tfrac{1}{2})$$

$$= x^2 - \frac{1}{3} - \frac{\frac{1}{12}}{\frac{1}{12}} (x - \tfrac{1}{2})$$

$$= x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6} = \frac{1}{6}\left(6x^2 - 6x + 1\right)$$

$$\int_0^1 \left(x - \frac{1}{2}\right)\left(x^2 - x + \frac{1}{6}\right) dx = \left\langle x - \frac{1}{2},\ x^2 - x + \frac{1}{6} \right\rangle = 0 \ !$$

$$u_1 = \frac{v_1}{\|v_1\|} = 1 \qquad u_2 = \frac{\left(x - \frac{1}{2}\right)}{\sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx}} \qquad \text{etc...}$$

Alternate Gram - Schmidt

Input : $w_1 \dots w_n$ a basis

Output : $\underline{u_1} \dots \underline{u_n}$ orthonormal basis

Actually we get more!

Assume

$$\begin{cases} \vec{w}_1 = r_{11}\vec{u}_1 \\ \vec{w}_2 = r_{12}\vec{u}_1 + r_{22}\vec{u}_2 \\ \vec{w}_3 = r_{13}\vec{u}_1 + r_{23}\vec{u}_2 + r_{33}\vec{u}_3 \\ \vdots \end{cases}$$

$$\vec{w}_n = r_{1n}\vec{u}_1 + r_{2n}\vec{u}_2 + \dots + r_{nn}\vec{u}_n$$

$v_1 = w_1$

$v_2 = w_2 - cv_1$

$$\begin{cases} w_2 = v_2 + cv_1 \\ w_3 = v_3 + c_1v_1 + c_2v_2 \end{cases}$$

$$\vec{w}_2 = r_{12}\vec{u}_1 + r_{22}\vec{u}_2 = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}\begin{pmatrix} r_{12} \\ r_{22} \end{pmatrix}$$

2 columns

$$\vec{w}_3 = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}\begin{pmatrix} r_{13} \\ r_{23} \\ r_{33} \end{pmatrix}$$

3 columns

$$w_1 = r_{11}u_1 + 0u_2 + \dots \quad 0u_n$$

$$w_2 = r_{12}u_1 + r_{22}u_2 + 0u_3 + \dots + 0u_n$$

$$w_3 = r_{13}u_1 + r_{23}u_2 + r_{33}u_3 + 0u_4 + \dots + 0u_n$$

$$\vdots$$

$\vdots$

$$w_1 = r_{11}u_1 + 0u_2 + \cdots\cdots 0u_n$$
$$w_2 = r_{12}u_1 + r_{22}u_2 + 0u_3 + \cdots + 0u_n$$
$$w_3 = r_{13}u_1 + r_{23}u_2 + r_{33}u_3 + 0u_4 + \cdots + 0u_n$$
$$\vdots$$
$$w_n = r_{1n}u_1 + \cdots\cdots\cdots\cdots r_{nn}u_n$$

$$\underbrace{\begin{pmatrix} \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{pmatrix}}_{\text{Input}} = \underbrace{\begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{pmatrix}}_{\text{Output: Orthonormal basis as columns}} \underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ & r_{22} & r_{23} & \cdots & \vdots \\ 0 & & r_{33} & \cdots & \vdots \\ & & & & r_{nn} \end{pmatrix}}_{\text{Upper }\Delta\text{ matrix of coefficients}}$$

1 known

2 unknowns ??

<u>Start</u>: $\vec{w}_1 = r_{11}\vec{u}_1 \longrightarrow u_1 = \dfrac{w_1}{\|w_1\|}$

$\vec{w}_2 = r_{12}\vec{u}_1 + r_{22}\vec{u}_2$

$\vec{w}_3 = r_{13}\vec{u}_1 + r_{23}\vec{u}_2 + r_{33}\vec{u}_3$

$\vdots$

$\vec{w}_n = r_{1n}\vec{u}_1 + r_{2n}\vec{u}_2 + \dots + r_{nn}\vec{u}_n$

<u>Step</u> 1 $\quad r_{11} = \|w_1\|$, $u_1 = \dfrac{\vec{w}_1}{r_{11}} = \dfrac{w_1}{\|w_1\|}$ $\quad$ u_1 known!

If u_1, u_2 are orthonormal then

$\langle w_2, u_1\rangle = \langle r_{12}u_1 + r_{22}u_2, u_1\rangle = r_{12}\overset{1}{\cancel{\langle u_1, u_1\rangle}} + r_{22}\overset{0}{\cancel{\langle u_1, u_2\rangle}}$

$r_{12} = \langle \underline{w_2}, \underline{u_1}\rangle$ $\quad$ known!

$\underbrace{w_2 = r_{12}u_1}_{\text{known}} + r_{22}\underline{u_2}$

Eq'n 4.8 If you have an orthonormal basis

$$v = c_1 u_1 + \dots + c_n u_n$$

$$\|v\| = \sqrt{c_1^2 + c_2^2 + \dots + c_n^2} \quad \text{for all } \langle -, - \rangle.$$

$$w_2 = r_{12} u_1 + r_{22} u_2 \implies \|w_2\| = \sqrt{r_{12}^2 + r_{22}^2}$$

$$r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2} \quad \text{known!}$$

$$u_2 = \frac{w_2 - r_{12} u_1}{r_{22}}$$

Step 2

- $r_{12} = \langle w_2, u_1 \rangle$
- $r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2}$
- $u_2 = \dfrac{w_2 - r_{12} u_1}{r_{22}}$

$$w_3 = r_{13} u_1 + r_{23} u_2 + r_{33} u_3$$

Step 3

comes from what we know about orthonormal bases

- $r_{13} = \langle w_3, u_1 \rangle$ *
- $r_{23} = \langle w_3, u_2 \rangle$ *
- $r_{33} = \sqrt{\|w_3\|^2 - r_{13}^2 - r_{23}^2}$ *
- $u_3 = \dfrac{w_3 - r_{13} u_1 - r_{23} u_2}{r_{33}}$

⋮

Step n

- $r_{in} = \langle w_n, u_i \rangle \qquad i < n$
- $r_{nn} = \sqrt{\|w_n\|^2 - r_{1n}^2 - \dots - r_{n-1,n}^2}$
- $u_n = \dfrac{w_n - r_{1n} u_1 - \dots - r_{n-1,n} u_{n-1}}{r_{nn}}$.

□

Ex $w_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $w_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $w_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ w/r/t dot product

Step 1

- $r_{11} = \|w_1\| = \sqrt{1^2 + 1^2 + (-1)^2} = \boxed{\sqrt{3}}$ $w_1 = r_{11} u_1$

- $u_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Step 2 $w_2 = r_{12} u_1 + r_{22} u_2$

- $r_{12} = \langle w_2, u_1 \rangle$

- $r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2}$

- $u_2 = \dfrac{w_2 - r_{12} u_1}{r_{22}}$

$$r_{12} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{3}} (1 - 2) = \boxed{\frac{-1}{\sqrt{3}}}$$

$r_{22}\left(= \cancel{\langle w_2, u_2 \rangle}\right)$ Would we ... $= \sqrt{\|w_2\|^2 - r_{12}^2}$

unknown

$$= \sqrt{5 - \frac{1}{3}} = \boxed{\sqrt{\frac{14}{3}}}$$

$$u_2 = \frac{\begin{pmatrix}1\\0\\2\end{pmatrix} - \left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}\right)}{\sqrt{\frac{14}{3}}} = \frac{\begin{pmatrix}1\\0\\2\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}}{\sqrt{\frac{14}{3}}}$$

$$= \boxed{\frac{1}{\sqrt{42}}\begin{pmatrix}4\\1\\5\end{pmatrix}}$$

Step 3

- $r_{13} = \langle w_3, u_1 \rangle$
- $r_{23} = \langle w_3, u_2 \rangle$
- $r_{33} = \sqrt{\|w_3\|^2 - r_{12}^2 - r_{13}^2}$

$w_3 = r_{13}u_1 + r_{23}u_2 + r_{33}u_3$

$$u_3 = \frac{w_3 - r_{13} u_1 - r_{23} u_2}{r_{33}}$$

$$r_{13} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \boxed{-\sqrt{3}}$$

$$r_{23} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{42}} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} = \boxed{\sqrt{\frac{21}{2}}}$$

$$r_{33} = \sqrt{17 - 3 - \frac{21}{2}} = \boxed{\sqrt{\frac{7}{2}}}$$

$$u_3 = \frac{\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - (-\sqrt{3})\left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right) - \sqrt{\frac{21}{2}} \, \frac{1}{\sqrt{42}} \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}}{\sqrt{\frac{7}{2}}}$$

$$= \boxed{\frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} \\ -\frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} & \frac{1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{-1}{\sqrt{3}} & -\sqrt{3} \\ 0 & \sqrt{\frac{14}{3}} & \sqrt{\frac{21}{2}} \\ 0 & 0 & \sqrt{\frac{7}{2}} \end{pmatrix} \quad !!$$

orthonormal basis

Eq'n (4.5) page 188 part (e) $u_1, \ldots, u_n$ orthonormal

$$\|v\| = \sqrt{c_1^2 + \ldots + c_n^2} \qquad v = c_1u_1 + \ldots + c_nu_n$$

Eq'n part (c)

(4.8) page 189 $v_1, \ldots, v_n$ orthogonal

$$\|v\|^2 = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\|v_i\|^2} = \frac{\langle v, v_1 \rangle}{\|v_1\|^2} + \frac{\langle v, v_2 \rangle}{\|v_2\|^2} + \cdots + \frac{\langle v, v_n \rangle}{\|v_n\|^2}$$

Verify