Exam 1 study material on canvas?

- 2 previous exams
- study guide/ list of topics
- 12/13 practice problems

$T(\vec{x}) = A\vec{x}$, given a matrix A $m \times n$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\Longrightarrow$ Four Fundamental Subspaces of A

$\vec{x} \longrightarrow A\vec{x}$

How?

a way to understand what we've been doing w/ row red and vector spaces

$$\begin{array}{cccc} * & & * & \\ \end{array}$$
$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & * & 0 & \boxed{*} \\ 0 & 0 & 1 & \boxed{*} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

independent

dependencies

Def let A be a $m \times n$ matrix $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$A = (\vec{v}_1 \dots \vec{v}_n)$

1) The kernel of A is the subspace

$$\ker(A) = \{ \vec{x} \mid A\vec{x} = 0 \} \subseteq \mathbb{R}^n$$

$=$ solutions to homogeneous system

If $x = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ $A\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0$

$\ker(A) =$ set of linear dependencies between the columns of A.

$A\vec{0} = \vec{0}$ ✓

$(Ax + Ay$
$0 + 0 = 0)$ ✓

$A(c\vec{x}) =$ ✓
$cAx = 0$

2) The image of A is the subspace

$$\operatorname{img}(A) = \{ v \mid A\vec{x} = \vec{v} \} \subseteq \mathbb{R}^m$$

(there exists $\vec{x}$)

$=$ set of vectors such that v is A times something

$\Rightarrow A\vec{x} = c_1 v_1 + \cdots + c_n v_n$ "column space" of A

so $\operatorname{img}(A) = $ Span of columns $=$ range of A

A^T is $n \times m$

3) The cokernel of A is

$$\operatorname{coker}(A) = \ker(A^T) \subseteq \mathbb{R}^m$$

$$= \{\vec{y} \mid A^T\vec{y} = 0\} = \left\{ \text{row vectors } \vec{y}^T A = 0 \right\}$$

4) The coimage of A is

$$\operatorname{coimg}(A) = \operatorname{img}(A^T) = \text{span of columns of } A^T$$

$$= \text{rows of } A = \text{"row space" of } A$$

$$\subseteq \mathbb{R}^n$$

$A : \mathbb{R}^n \to \mathbb{R}^m$

$A^T : \mathbb{R}^m \to \mathbb{R}^n$

They're subspaces of $\mathbb{R}^n, \mathbb{R}^m$ so they have a dimension!

$\dim \operatorname{img}(A) = \dim(\text{span of columns of } A)$

$= \#$ of independent columns

$\dim(\operatorname{coimg}(A)) = \#$ of independent rows

$\dim(\ker(A)) = \#$ of free columns in RREF. (why?)

Thm

$\dim(\ker(A)) = \#$ of free column of RREF of A

$\dim(\operatorname{img}(A)) = \#$ of leading 1's in RREF of A

(we'll answer why independent columns of $A \leftrightarrow$ leading 1's)

$\|$

$\dim(\operatorname{coimg}(A)) = \#$ of leading 1's in RREF of A

$\dim(\operatorname{coker}(A)) = \#$ of rows of 0's in RREF of A

Pf

1) Claim: Row operations

i) preserve the kernel and coimage of A

$A \xrightarrow[\text{op}]{\text{row}} A' \implies \ker(A) = \ker(A')$

$\implies \ker(A) = \ker(RREF(A))$

$\implies \text{coimg}(A) = \text{coimg}(RREF(A))$.

why? row ops are elementary matrices!

$A \xrightarrow[\text{op}]{\text{row}} A'$, $A' = EA$

elementary matrix

$\ker(A') = \ker(EA) = \{\vec{x} \mid EA\vec{x} = 0\}$

$= \{\vec{x} \mid A\vec{x} = E^{-1}0 = 0\}$

$= \ker(A)$

$$\mathrm{coimg}(A') = \mathrm{coimg}(EA) = \{c_1 r_1 + \dots + c_j(c r_i + r_j) + c_m r_m\}$$

$$= \{c_1 r_1 + \dots + (c_i + c_j c)(r_i) + \dots + c_j r_j + \dots + c_m r_m\}$$

$$= \mathrm{coimg}(A)$$

$$\ker(A) = \ker(RREF) = \underbrace{x_1 \begin{pmatrix} \vdots \end{pmatrix} + \dots + x_k}_{\text{free variables}} \begin{pmatrix} \vdots \end{pmatrix}$$

2) $\dim(\ker(A)) = \#$ free variables

$\dim(\mathrm{coimg}(A)) = \dim(\mathrm{coimg}(RREF)) = \#$ of independent rows of RREF

$= \#$ of rows w/ leading 1

$$\begin{pmatrix} 1 & 0 & * & * \\ & 1 & * & * \end{pmatrix} \quad \times$$

$= \#$ of leading 1's.

ii)

So if $A \xrightarrow[\text{op}]{\text{row}} A'$, $\operatorname{Img}(A) \neq \operatorname{Img}(A')$. (different from ker)

But!

2) $\dim(\operatorname{img}(A)) \overset{\text{why?}}{=} \dim(\operatorname{img}(A')) = \dim(\operatorname{img}(\text{RREF}))$

since $\ker(A) = \ker(A')$

$= \#$ independent columns of RREF

$= \#$ of leading 1's

$$\begin{array}{c} \begin{pmatrix} 1 & * & 0 & * & c_1 \\ 0 & & 1 & * & c_2. \\ 0 & & 0 & & \\ 0 & & 0 & & \end{pmatrix} \\ \quad e_1 \quad\; e_2 \end{array}$$

$c_1 v_1 + c_2 v_2 = v_5 \longleftarrow c_1 e_1 + c_2 e_2 = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} v_1 & v_2 & v_3 & w \end{pmatrix} \longrightarrow \begin{pmatrix} v_1' & v_2' & v_3' & w' \end{pmatrix}$$

$$w' = c_1 v_1' + c_2 v_2' + c_3 v_3'$$

$$w = c_1 v_1 + c_2 v_2 + c_3 w_3$$

$$\Downarrow$$

$$\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ -1 \end{pmatrix} \in \ker(A) \longrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ -1 \end{pmatrix} \in \ker(A')$$

<u>Dependencies</u> are <u>preserved</u> by row <u>operations</u>!

Dependencies <u>of RREF tell us dependencies of A.</u>

$\square$

Fundamental Theorem of Linear Algebra / Rank-Nullity Theorem:

A $\underline{m \times n}$

1) $\dim \ker(A) = \#$ of free columns

$\dim(\operatorname{img}(A)) = \#$ of leading 1's

Rank Nullity thm

$$\underbrace{\dim(\ker(A))}_{= \text{nullity}(A)} + \underbrace{\dim(\operatorname{img}(A))}_{= \text{rank}(A)} = \#\text{ of columns} = n$$

If A is $\boxed{n \times n}$, we can tie this into what we know about invertible matrices from section 1.

Remember A^{-1} exists $\iff A \longrightarrow I$

$I = \text{RREF}(A)$.

Fundamental Theorem of Linear Algebra

The following are equivalent. (If any is false ⇒ all false. If any is true ⇒ all true.)

(changes depending on what matrix I give you)

A $n \times n$

1) A^{-1} exists
2) $A \longrightarrow I$ by row reduction
3) A has n leading 1's i.e. rank$(A) = n$
4) $\text{img}(A) = \mathbb{R}^n$ (span) (range = codomain)
5) $\ker(A) = 0$ (no free columns) (independent)
6) columns of A form a basis of $\mathbb{R}^n$
7) A is a permuted LU decomposition
8) $\det(A) \neq 0$.

we learned this today!

OH: Thursday at 12pm!