


Exam 1 study material on canvas!

- 2 previous exams
- study guide/ list of topics
- 12/13 practice problems

$T(\vec{x}) = A\vec{x}$, given a matrix A $m \times n$

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$
$$\vec{x} \rightarrow A\vec{x}$$

Four Fundamental Subspaces of A

How?

$$\begin{bmatrix} * & * \\ v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

independent

$$\begin{bmatrix} * & * \\ 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

dependencies

a way to understand what we've been doing w/ row red and vector spaces

Def Let A be a $m \times n$ matrix $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$
 $A = (\vec{v}_1, \dots, \vec{v}_n)$

1) The kernel of A is the subspace

$$\ker(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n$$

= solutions to homogeneous system

$$\text{If } \vec{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad A\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$$

$\ker(A)$ = set of linear dependencies between the columns of A .

$$A\vec{0} = \vec{0} \quad \checkmark$$

$$(A\vec{x} + A\vec{y} = \vec{0}) \quad \checkmark$$
$$0 + 0 = 0$$

2) The image of A is the subspace

$$\text{img}(A) = \left\{ \vec{v} \mid A\vec{x} = \vec{v} \right\} \subseteq \mathbb{R}^m$$

(there exists \vec{x})

$$A(c\vec{x}) = cA\vec{x} = \vec{0} \quad \checkmark$$

= set of vectors such that \vec{v} is A times something

$$\Rightarrow A\vec{x} = a_1v_1 + \dots + a_nv_n \quad \text{"column space" of } A$$

$$\text{so } \text{Im}(A) = \underline{\text{Span of columns}} = \text{range of } A$$

3) The cokernel of A is A^T is $n \times m$

$$\text{coker}(A) = \ker(A^T) \subseteq \mathbb{R}^m$$

$$= \left\{ \vec{y} \mid A^T \vec{y} = 0 \right\} = \left\{ \text{row vectors } \vec{y}^T A = 0 \right\}$$

4) The wimage of A is

$$\text{wim}(A) = \text{Im}(A^T) = \text{span of columns of } A^T$$

$$= \text{rows of } A = \text{"row space" of } A$$

$$\subseteq \mathbb{R}^n$$

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$A^T: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

They're subspaces of $\mathbb{R}^n, \mathbb{R}^m$ so they have a

dimension!

$$\underline{\dim \operatorname{Im}(A)} = \dim(\operatorname{span} \text{ of columns of } A)$$

$$= \# \text{ of independent columns}$$

$$\dim(\operatorname{coIm}(A)) = \# \text{ of independent rows}$$

$$\dim(\ker(A)) = \# \text{ of free columns in RREF. (Why?)}$$

Thm $\dim(\ker(A)) = \# \text{ of free columns of RREF of } A$

$$\dim(\operatorname{Im}(A)) = \# \text{ of leading 1's in RREF of } A$$

|| $(\text{we'll answer why independent columns of } A \leftrightarrow \text{leading 1's})$

$$\dim(\operatorname{coIm}(A)) = \# \text{ of leading 1's in RREF of } A$$

$$\dim(\operatorname{coker}(A)) = \# \text{ of rows of 0's in RREF of } A$$

Pf

1) Claim: Row operations

i) preserve the kernel and coimage of A

$$A \xrightarrow[\text{op}]{\text{row}} A' \implies \ker(A) = \ker(A')$$

$$\implies \ker(A) = \ker(\text{REF}(A))$$

$$\implies \text{coimg}(A) = \text{coimg}(\text{REF}(A)).$$

why? row ops are elementary matrices!

$$A \xrightarrow[\text{op}]{\text{row}} A', \quad A' = EA$$

↑ elementary matrix

$$\begin{aligned} \underline{\ker(A')} &= \ker(EA) = \{ \vec{x} \mid EA\vec{x} = 0 \} \\ &= \{ \vec{x} \mid A\vec{x} = E^{-1}0 = 0 \} \\ &= \underline{\ker(A)} \end{aligned}$$

$$\begin{aligned}
 \underline{\text{Colng}(A')} &= \text{Colng}(EA) = \{c_1 r_1 + \dots + c_j (c_i r_i + r_j) + c_m r_m\} \\
 &= \{c_1 r_1 + \dots + (c_i + c_j c) r_i + \dots + r_j \\
 &\quad + \dots + c_m r_m\} \\
 &= \underline{\text{Colng}(A)}
 \end{aligned}$$

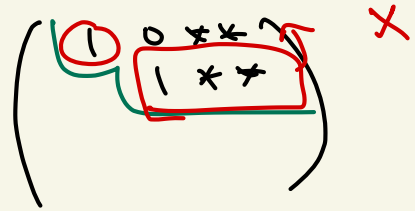
$$\text{Ker}(A) = \text{Ker}(\text{REF}) = \underbrace{x_1 \begin{pmatrix} i \\ \vdots \end{pmatrix} + \dots + x_k \begin{pmatrix} i \\ \vdots \end{pmatrix}}_{\text{free variables}}$$

$$\hookrightarrow \dim(\text{Ker}(A)) = \# \text{ free variables}$$

$$\dim(\text{Colng}(A)) = \dim(\text{Colng}(\text{REF})) = \# \text{ \textit{v} independent rows of REF}$$

= # of rows w/ leading 1

= # of leading 1's.



ii)

So if $A \xrightarrow[\text{up}]{\text{row}}$ A' , $\text{img}(A) \neq \text{img}(A')$. (different from ker)

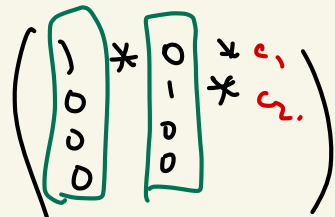
But!

2) $\dim(\text{img}(A)) \stackrel{\text{why?}}{=} \dim(\text{img}(A')) = \dim(\text{img}(\text{RREF}))$

= # independent columns of RREF

= # of leading 1's

Since $\ker(A) = \ker(A')$



$c_1 v_1 + c_2 v_2 = v_5$

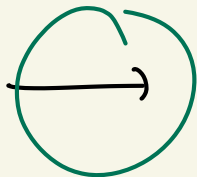
$c_1 e_1 + c_2 e_2 = \begin{pmatrix} c_1 \\ c_2 \\ 0 \\ 0 \end{pmatrix}$

$$(v_1, v_2, v_3, w) \longrightarrow (v_1', v_2', v_3', w')$$

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$w' = c_1 v_1' + c_2 v_2' + c_3 v_3'$$

$$\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \in \ker(A)$$



$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{pmatrix} \in \ker(A')$$

Dependencies are preserved by row operations!

Dependencies of RREF tell us dependencies of A.

□

Fundamental Theorem of Linear Algebra

Rank-Nullity Theorem:

$$A \quad \underline{m \times n}$$

1) $\dim \ker(A) = \#$ of free columns

$\dim(\text{img}(A)) = \#$ of leading 1's

$$\dim(\ker(A)) + \dim(\text{img}(A)) = \# \text{ of columns} = n$$

Rank
Nullity
thm

"
nullity(A)

"
rank(A)

If A is $\boxed{n \times n}$ we can tie this into what we know about invertible matrices from section 2.

Remember A^{-1} exists $\iff A \rightarrow I$

$$I = \text{REF}(A).$$

Fundamental Theorem of Linear Algebra

The following are equivalent.

A $n \times n$

1) A^{-1} exists

2) $A \rightarrow I$ by row reduction

3) A has n leading 1's i.e. $\text{rank}(A) = n$

4) $\text{img}(A) = \mathbb{R}^n$ (span) (range = codomain)

5) $\text{ker}(A) = 0$ (no free columns) (independent)

6) columns of A form a basis of \mathbb{R}^n

7) A is a permuted LU decomposition

8) $\det(A) \neq 0$

(If any is false \Rightarrow all false.)

If any is true \Rightarrow all true.

(changes depending on what matrix I give you)

we learned this today!

OH! Thursday at 12pm!