Exam 1 - Friday 10/9.

- Email w/ exam info later day (also in monday's lecture notes)

- solutions to study guide problems later today also

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4 fundamental subspaces of a matrix A, $m \times n$

$\ker(A)$ | $:= \{\vec{x} \mid A\vec{x} = 0\}$ = set of solutions to $Ax = 0$

(compute by RREF / row reduction)

= set of dependencies between columns of A

$$A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Then $A\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

If $\vec{x} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \ker(A) \iff c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$

i.e. a linear relationship between columns!

$Ax=0$ has solutions from RREF

$\implies \dim(\ker(A)) = \#$ of free variables

A $m \times n$

$\text{img}(A) = \{ v \mid Ax = v \} \subseteq \mathbb{R}^m$

= all of the vectors $\vec{v}$ such that

$A\vec{x} = \vec{v}$ has a solution

$=$ span of columns of A

$\dim(\mathrm{img}(A))$ = size of a basis of the span of columns of A

pick out independent vectors

$= \dim(\mathrm{span}(v_1, \ldots, v_n)) =$

$=$ # of independent columns

$=$ column w/ leading 1's = # of leading 1's.

$= \underline{\mathrm{rank}(A)}$

Every column in RREF is either free or has a leading 1.

$\Rightarrow$ rank-nullity theorem

$\dim(\ker(A)) + \dim(\mathrm{img}(A)) = n$

** # of free vars + # of leading 1's = total # of columns.

** important!

matrix need not be square!

$m \times n$

$\text{coimg}(A) = \text{img}(A^T) = \text{span of rows of } A$

$\dim(\text{coimg}(A)) = \#$ of leading 1's of A $= \dim(\text{coimg}(A))$

$\text{rank}(A^T) = \text{rank}(A)$

Even though RREF(A) and RREF(A^T) have the same number of leading 1's, they won't be the same exact matrix.

neat

$\text{coker}(A) = \ker(A^T)$, $\dim(\text{coker}(A)) = \#$ of rows of 0's in RREF(A).

will be neat

Solve

$$\begin{aligned} 2x - y - 5z + w + u &= 0 \\ x + 3y + z + w + 2u &= 0 \\ 2x + 0y - 4z + 0w - 2u &= 0 \end{aligned} \longrightarrow$$

$$\underbrace{\begin{pmatrix} 2 & -1 & -5 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 0 & -4 & 0 & -2 \end{pmatrix}}_{3\times 5} \underbrace{\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix}}_{5\times 1} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{3\times 1}$$

RREF not only will compute the solution set. we can figure out a lot more.

$$\begin{pmatrix} 2 & -1 & -5 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 0 & -4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{pmatrix} \boxed{1} & 0 & -2 & 0 & -1 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{pmatrix} \\ \begin{array}{ccccc} & & \text{free} & & \text{free} \\ x & y & z & w & u \end{array} \end{array}$$

$$\begin{matrix} \parallel \\ B \end{matrix}$$

$$x = 2z + u$$

$$y = -z \quad \rightsquigarrow$$

$$w = -3u$$

$$\ker(B) = \text{solution set to } B\vec{x} = \vec{0}$$

$$= \begin{pmatrix} 2z + u \\ -z \\ z \\ -3u \\ u \end{pmatrix} = \begin{pmatrix} 2z \\ -z \\ z \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u \\ 0 \\ 0 \\ -3u \\ u \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix} u \right\}$$

$$\ker(B) = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

All along we could have computed ker(B) as a span.

These vectors are independent, they span the kernel

$$\Longrightarrow \quad z\ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix}\ u$$ form a basis of ker(B).

(This always works!)

$\dim(\ker(A)) = \#$ basis elements of $\ker(A)$ $= 2$

$\dim(\ker(A)) = \#$ free variables $= 2$

$$\begin{pmatrix} 2 & -1 & -5 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 0 & -4 & 0 & -2 \end{pmatrix}$$

Which columns are independent?

Which ones depend on the others?

Ker(B) is the set of dependencies (linear relationships between columns)

$$\begin{pmatrix} 2 & -1 & -5 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 0 & -4 & 0 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + 1\underbrace{\begin{pmatrix} -5 \\ 1 \\ -4 \end{pmatrix}}_{} = 0$$

Solve for this vector

Since $\ker(B) = \ker(RREF(B))$, the same linear relationships hold between the columns.

$$\begin{array}{ccccc} * & * & & * & \\ \end{array} \qquad\qquad \begin{array}{ccccc} * & * & * & * & * \end{array} \quad \text{rank}(B) + \dim(\ker(B))$$

$$\begin{pmatrix} 2 & -1 & -5 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 \\ 2 & 0 & -4 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & -2 & 0 & -1 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 \end{pmatrix} = 5$$

$$\underset{3^{rd}\text{ col}}{\begin{pmatrix} -5 \\ 1 \\ -4 \end{pmatrix}} = -2 \underset{1^{st}\text{ col}}{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} + 1 \underset{2^{nd}\text{ col}}{\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}} \qquad \underset{3^{rd}}{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}} = -2 \underset{1^{st}}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + 1 \underset{2^{nd}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

3^{rd} column depends on first 2.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix} \in \ker(B)$$

$$\Longrightarrow \quad \underset{5^{th}}{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}} = -1 \underset{1^{st}}{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} + 3 \underset{4^{th}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

basis of $\mathrm{img}(B)$

$\mathrm{rank}(B)$ = # of independent columns

$$= \# \left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$= 3$ = # of leading 1's = $\dim(\mathrm{img}(B))$

$\dim(\ker(B)) = 2$

$2 + 3 = 5$ = # of columns

Thm The following are equivalent! (A is $n \times n$)

1) A^{-1} exists
2) $A \longrightarrow I$ ← n 1's
3) permuted LU decomp
4) $\det(A) \neq 0$
5) $\text{rank}(A) = n$
6) $\ker(A) = 0$
7) Columns form a basis of $\mathbb{R}^n$

$$(A \mid I) \longrightarrow (I \mid A^{-1})$$

(# columns − leading 1's = dim ker(A))

$$n - n = 0$$

* 8) rows form a basis of $\mathbb{R}^n$ *

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{swap } r_1 r_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}r_2 + r_3} \quad \text{etc}$$

$$\vdots$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & -3 & 2 & 1 \\ 0 & \boxed{1} & 0 & -3 & 2 & 0 \\ 0 & 0 & \boxed{1} & 2 & -1 & 0 \end{array}\right)$$

RREF of A

A^{-1}

A^{-1} exists

$\Updownarrow$

3 leading 1's

$\Updownarrow$

$A \to I$

$\Updownarrow$

$\text{rank}(A) = 3 = \#$ of columns

$\Updownarrow$

columns are independent

$\Updownarrow$

basis of $\mathbb{R}^3$!

$\text{nullity} = \dim(\ker(A))$

$= \dim\{0\} = 0.$

$\Longleftrightarrow \ker(A) = 0$ (no free variables!)

$3 + 0 = 3$ (rank + nullity = # of columns)

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 5 & 7 \\ 0 & 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

depends on v_1, v_2

A^{-1} doesn't exist!

$V = \mathbb{R}^2$ $\qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a basis of $\mathbb{R}^2$

If $\quad c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies c_1, c_2 = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So independent!

Every vector $\begin{pmatrix}x\\y\end{pmatrix} \in \text{span}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right)$. they span $\mathbb{R}^2$

Are all systems $\underbrace{\begin{pmatrix}1&1\\1&-1\end{pmatrix}}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ consistent?

Do they have a solution?

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}=\frac{1}{\det}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$$

$$=\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$=\frac{1}{1(-1)-(1)(1)}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$$

$$c_1=\frac{1}{2}x+\frac{1}{2}y$$

$$c_2=\frac{1}{2}x-\frac{1}{2}y$$

$$=\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$\left(\frac{1}{2}x+\frac{1}{2}y\right)\begin{pmatrix}1\\1\end{pmatrix}+\left(\frac{1}{2}x-\frac{1}{2}y\right)\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

$\begin{pmatrix}x\\y\end{pmatrix}\in\text{span}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}$ span $\mathbb{R}^2$.

Every basis of $\mathbb{R}^4$ has 4 vectors

$$\begin{pmatrix}1\\3\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\0\end{pmatrix}$$

Could never span $\mathbb{R}^4$!

not enough dimensions

span $\neq \mathbb{R}^4$, not span

$$\begin{pmatrix}1\\3\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\0\end{pmatrix}, \begin{pmatrix}5\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\2\\2\end{pmatrix}$$

Could never be independent

$\downarrow$

at best

$$\begin{pmatrix}1 & & & & *\\ & 1 & & & *\\ & & 1 & & *\\ & & & 1 & *\end{pmatrix}$$

5th vector depends on other 4

Image of a matrix $\quad A = (\vec{v}_1 \cdots \vec{v}_n)$

$\mathrm{img}(A) = \mathrm{span}(\text{columns of } A)$ know this!

$\dim(\mathrm{img}(A)) = \#$ of independent columns

$$\mathrm{img}(A) = \mathrm{span}(\text{columns}) = \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n\}$$ $c_1 \cdots c_n$ varying

$$= \left\{ (\vec{v}_1 \cdots \vec{v}_n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right\} \qquad \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \vec{x}$$ varying

vector of variables

$$= \{ v = A\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \}$$

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\vec{x} \longrightarrow A\vec{x}$$

$\text{img}(A)$ = range of this function

all possible outputs of multiplying by A.

$\text{coimg}(A) = \text{img}(A^T)$ = span columns of A^T

= span of rows of A

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

$\text{span}\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \right)$

$= \text{coimg}(A)$

= span of rows of A