Exam Friday!

1) Given a $\text{span}(\vec{v}_1 \dots \vec{v}_k)$, compute $\text{span}(\vec{v}_1, \dots, \vec{v}_k)^{\perp}$

(4.4.12b)

Ex $\text{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right)$

If $\vec{x} \perp \boxed{\text{span}\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}} \implies \vec{x} \perp \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}$

$$\implies \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \vec{x} = 0$$
$$\begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \vec{x} = 0$$

$$\implies \boxed{\begin{pmatrix}1 & 2 & 3\\ 2 & 0 & 1\end{pmatrix}\vec{x} = \begin{pmatrix}0\\0\end{pmatrix}}$$ Linear system!

$\xrightarrow[\text{to RREF}]{\text{row reduce}}$ $\begin{pmatrix}\boxed{1} & 0 & \frac{1}{2}\\ 0 & \boxed{1} & \frac{5}{4}\end{pmatrix}$ free!

leading 1's

$x = -\frac{1}{2}z$

$y = -\frac{5}{4}z$

$$\text{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}\right)^{\perp} = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix},\ x=-\frac{1}{2}z,\ y=-\frac{5}{4}z\right\}$$

$$= \left\{\begin{pmatrix}-\frac{1}{2}z\\-\frac{5}{4}z\\z\end{pmatrix}\right\} = \text{Span}\begin{pmatrix}-\frac{1}{2}\\-\frac{5}{4}\\1\end{pmatrix}$$

$$\left(= \text{Span}\left(\frac{1}{4}\begin{pmatrix}-2\\-5\\4\end{pmatrix}\right)\right)$$

$$= \text{Span}\begin{pmatrix}-2\\-5\\4\end{pmatrix}$$

In theory $\begin{pmatrix}-2\\-5\\4\end{pmatrix} \perp \begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\0\\1\end{pmatrix}$

$$(-2,-5,4)\cdot(1,2,3) = -2-10+12=0$$

$$(-2,-5,4)\cdot(2,0,1) = -4+0+4=0$$

Optional

But suppose $\text{span}(v_1 \dots v_k)$ k independent vectors in $\mathbb{R}^n$

then $W^\perp = \text{span}(\vec{x}_1 \dots x_{n-k})$ $n-k$ basis vectors for $W^\perp$.

$n=k$
$\Rightarrow W^\perp = \{0\}$

$\text{span}\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}$ $\boxed{2}$ independent vectors $\mathbb{R}^{\boxed{3}}$

$W^\perp = $ span of $3-2$ vectors, aka 1 vector.

$$\text{span}\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{matrix}1&2&3\\2&0&1\\0&0&1\end{matrix} \xrightarrow{\text{RREF}} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So $\text{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)^\perp = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

$\boxed{2)}$ Prop If $W \subseteq V$ then $W \cap W^\perp = \{0\}$.

Idea: $\vec{0}$ vector is the only vector perpendicular to itself!

3) Thm Given a matrix A $m \times n$. Then

- $\ker(A) = \text{coimg}(A)^\perp$

all solutions to $A\vec{x} = \vec{0}$

span of rows of A = span of columns of A^T = $\text{img}(A^T)$

Idea: rows of A $\perp$ ker of A

$$\underset{A}{\begin{pmatrix} - r_1 - \\ - r_m - \end{pmatrix}} \underset{x}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} \underset{= 0}{=} \vec{0} \longrightarrow \begin{array}{l} \vec{r}_1 \cdot \vec{x} = 0 \\ \vec{r}_2 \cdot \vec{x} = 0 \\ \vdots \\ \vec{r}_m \cdot \vec{x} = 0 \end{array} \qquad \vec{r}_i \perp \vec{x}$$

- $\operatorname{coker}(A)^{\perp} = \operatorname{img}(A)$

solutions to

$A^T x = 0$

$\Updownarrow$

$x^T A = 0$

span of columns of A

Idea: coker $\perp$ columns of A

4.4.12b

$$\operatorname{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right)^{\perp} = \operatorname{coimg}\begin{pmatrix}1 & 2 & 3\\ 2 & 0 & 1\end{pmatrix}^{\perp}$$

thm!

$$= \ker\begin{pmatrix}1 & 2 & 3\\ 2 & 0 & 1\end{pmatrix} = \operatorname{Span}\begin{pmatrix}-2\\-5\\4\end{pmatrix}$$

4.4.29b

$$\begin{pmatrix}5 & 0\\ 1 & 2\\ 0 & 2\end{pmatrix}$$

Compute 4 Find subspaces, verify orthogonality thm.

- $\ker\begin{pmatrix}5 & 0\\ 1 & 2\\ 0 & 2\end{pmatrix}$ = Solutions to $\begin{pmatrix}5 & 0\\ 1 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

RREF!! $\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$

$$= \boxed{\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}} \quad *$$

- $\text{Img}\begin{pmatrix} 5 & 0 \\ 1 & 2 \\ 0 & 2 \end{pmatrix}$ = span of columns of A = $\boxed{\text{span}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}}$ *

- $\text{coker}(A) = \ker(A^T) = \ker\begin{pmatrix} 5 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \underset{\text{RREF}}{=} \boxed{\text{span}\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix}}$ *

- $\text{coimg}(A)$ = span of rows of A = $\text{span}\begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \boxed{\mathbb{R}^2}$ *

* basis of $\mathbb{R}^2$ *

$\text{coimg}\begin{pmatrix} 5 & 0 \\ 1 & 2 \\ 0 & 2 \end{pmatrix}$

$\ker(A) = \text{coimg}(A)^\perp.$

$\left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} = (\mathbb{R}^2)^\perp$ ✓✓

True $(\mathbb{R}^2)^\perp$

= vectors $\perp$ to everything

$= \{\vec{0}\}.$

$\text{coker}(A) = \text{img}(A)^\perp$

Claim: $\text{span}\begin{pmatrix} 1 \\ -5 \\ 5 \end{pmatrix} \perp \text{span}\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

$\begin{pmatrix}1\\-5\\5\end{pmatrix} \perp \begin{pmatrix}5\\1\\0\end{pmatrix}$ since $(1,-5,5)\cdot(5,1,0) = 5-5+0=0$ ✓

$\begin{pmatrix}1\\-5\\5\end{pmatrix} \perp \begin{pmatrix}0\\2\\2\end{pmatrix}$ since $(1,-5,5)\cdot(0,2,2) = 0-10+10=0$ ✓

The theorem works!

With functions it's complicated... not on syllabus

$W = P^{(\infty)}$ = all polynomials of any degree $\subseteq C^0[a,b]$

Turns out $\left(P^{(\infty)}\right)^{\perp} = \{0\}$ Reason: Taylor series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Also not on syllabus

$$V = \mathbb{R}^n, \quad \langle v, w \rangle = v^T K w$$

$W^\perp$ w/ $\langle v, w \rangle$??

(ONLY $W^\perp$ w/ dot product on exam!)

Maybe remember how to row reduce, + FS

but not cumulative

L^1, L^2, L^∞ on $\mathbb{R}^n$ or $C^0[a,b]$

Complex vector space (3.6) (Check out 3.6 HW problems!)

- Row reduce a complex matrix
- Fundamental thm of Linear alg still true for complex numbers!

$$\left(\begin{array}{ll} \cdot\ A^{-1} \text{ invertible} & \cdot\ A \longrightarrow I \\ \cdot\ A \text{ ind columns} & \cdot\ \ker(A) = 0 \\ \cdot\ A \text{ ind rows} & \cdot\ \operatorname{img}(A) = \mathbb{R}^n \\ \cdot\ \det(A) \neq 0 & \end{array} \right)$$

Office Hours: 12 - 3 pm tomorrow!