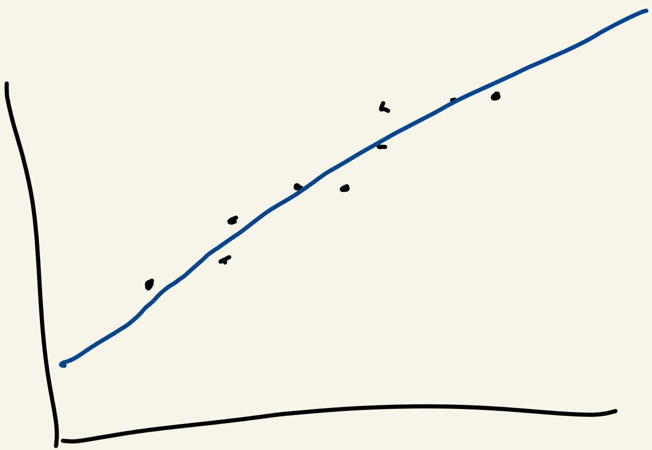
HW 8 available

Ch 5 : Linear algebra approach to linear regression



"best fit line"

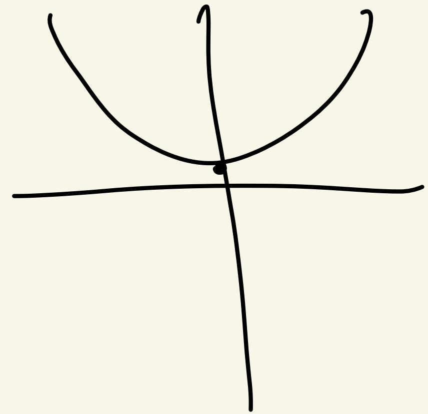
$\updownarrow$

minimization problem!

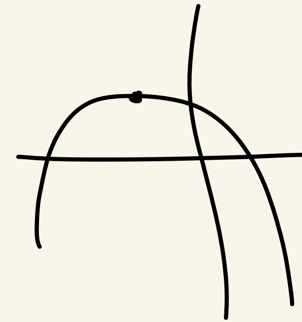
Section 5.2 Minimazation of quadratics (degree 2 polynomial in many variables)

1 variable

$$p(x) = ax^2 + bx + c$$

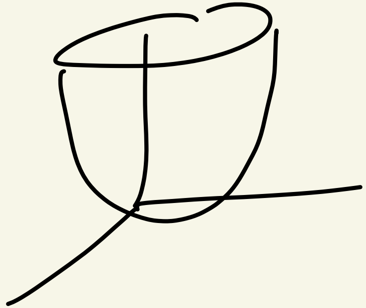


$\boxed{a>0}$ min

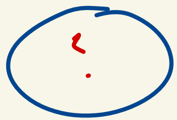


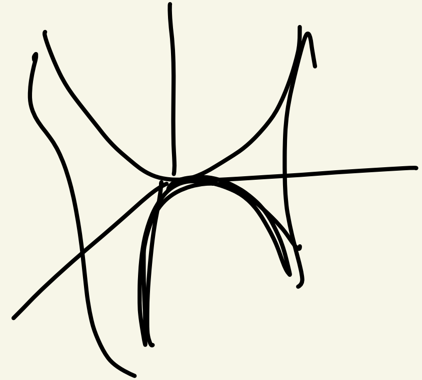
$\boxed{a<0}$ max

$$p(x,y) = \underbrace{\boxed{a}x^2 + \boxed{b}xy + \boxed{c}y^2}_{\text{deg } 2} + \underbrace{dx + ey}_{\text{deg } 1} + \underbrace{f}_{\text{constant}}$$



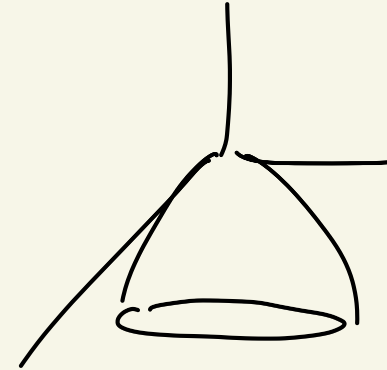
facing up





pringle shape
" saddle "

? no min



facing down

?
no min

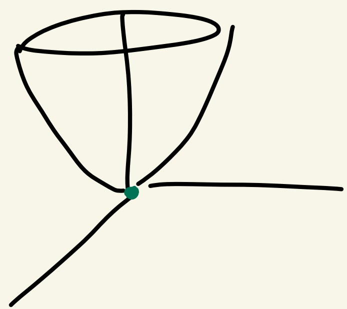
$$p(x,y) = \underbrace{\boxed{a}x^2 + \boxed{b}xy + \boxed{c}y^2}_{\text{quadratic form}} + dx + ey + f$$

$$x^T K x \quad + \quad x^T \vec{f} \quad + \quad c \quad (\text{constant})$$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \boxed{a} & \boxed{\frac{b}{2}} \\ \boxed{\frac{b}{2}} & \boxed{c} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix} + c$$

let $K = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

If K is pos def then $p(x,y)$ faces up and p has a minimum value!



So if $K > 0$ have a minimization problem! We'll stick to K being positive definite.

Problem: Given a quadratic w/ n variables

$$p(x_1, x_2, \ldots, x_n) = \sum_{i \le j}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c.$$

$x_i x_j = x_j x_i$

So I'm not adding this term twice

When does this have a min value? If so for $\vec{x}$ does it occur? What's the actual minimum value?

First : $p(\vec{x}) = \underbrace{\sum a_{ij} x_i x_j}_{x^T K x} + \sum \boxed{b_i} x_i + c \qquad \vec{x} = (x_1, \dots x_n)$

$\cancel{x^T \vec{b}}$

$-2x^T \vec{f} \quad + c$

$$K = \begin{pmatrix} a_{11} & & \frac{a_{ij}}{2} \\ & a_{22} & \\ \frac{a_{ij}}{2} & \ddots & \\ & & a_{nn} \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} \boxed{-b_1} \\ \overline{2} \\ \frac{-b_2}{2} \\ \vdots \\ \frac{-b_n}{2} \end{pmatrix}$$

$$p(\vec{x}) = \underbrace{(x_1 \dots x_n) \begin{pmatrix} a_{11} & & \frac{a_{ij}}{2} \\ & \ddots & \\ \frac{a_{ij}}{2} & & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} - 2(x_1 \dots x_n) \begin{pmatrix} \frac{-b_1}{2} \\ \vdots \\ \frac{-b_n}{2} \end{pmatrix}$$

$+ \; c$

$$p(\vec{x}) = x^T K x - 2x^T \vec{f} + c$$

the form of the quadratics that we'll use.

Claim: $p(\vec{x})$ has a min only when K is pos def.

The min occurs at $x^* = K^{-1} f$, and the min value is $p(x^*) = c - f^T x^*$.

Pf: Assume $K > 0$. If $x^* = K^{-1} f \Rightarrow f = K x^*$.

$$p(\vec{x}) = \overset{x^2}{x^T K x} \overset{-2bx}{- 2x^T (K x^*)} \overset{+b^2 - b^2}{+ c}$$

complete the square

$$= (x - x^*)^T K (x - x^*) + (c - x^{*T} K x^*)$$

$$x^2 \quad - \quad 2bx \quad + b^2 \quad - b^2 \longrightarrow (x-b)^2$$

$$= \left(x^T K x - 2x^T (K x^*) + x^{*T} K x^* \right) - \left(x^{*T} K x^* + c \right)$$

$$= (x - x^*)^T K (x - x^*) + (c - f^T x^*)$$

$$p(\vec{x}) = \underbrace{(x - x^*)^T K (x - x^*)} + (c - f^T x^*) \longrightarrow \text{constant!}$$

$K > 0$ so $q(\vec{y}) = y^T K y$ has a min at $y = 0$

i.e. $q(\vec{y}) = 0$ iff $\vec{y} = 0$!

$p(\vec{x})$ has a minimum only at $x - x^* = 0$

$\Longrightarrow$ x^* is the minimizer.

When $x = x^*$, the $* = 0 \Longrightarrow p(x^*) = c - f^T x^*$. $\square$

Ex!

$$p(x,y,z) = \underbrace{\boxed{1}x^2 + \boxed{2}xy + \boxed{\overset{1}{}}xz + \boxed{2}y^2 + \boxed{\overset{1}{}}yz + \boxed{2}z^2}\ + \underline{\boxed{6}y - \boxed{7}z} \overset{+0x}{} + 5$$

① $p(x,y,z) = x^T K x - 2x^T f + c$. $K = ?$ $f = ?$

$$K = \begin{array}{c|ccc} & x & y & z \\ x & 1 & 1 & \frac{1}{2} \\ y & 1 & 2 & \frac{1}{2} \\ z & \frac{1}{2} & \frac{1}{2} & 2 \end{array} \qquad f = \underline{\begin{pmatrix} 0 \\ -3 \\ 7/2 \end{pmatrix}} \qquad b = \begin{pmatrix} 0 \\ 6 \\ -7 \end{pmatrix}$$

K is symmetric by construction $K^T = K$ K^{-1} ??

$$p(\vec{x}) = (x\ y\ z)\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\,(x\ y\ z)\begin{pmatrix} 0 \\ -3 \\ 7/2 \end{pmatrix} + 5$$

K is pos def! So it has a min

Note: All problems on HW involve pos definite matrices.

our reduction

$$x^* = K^{-1} f$$

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ -3 \\ 7/2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

$p(x,y,z)$ has the lowest value at $(2,-3,2)$.

$$p(2,-3,2) = 5 - f^T x^* = 5 - \left(0\ \ -3\ \ 7/2\right)\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$= -11$$

Ex $p(x,y) = 4x^2 - 2xy + 3y^2 + 3x - 2y + 1$

$$= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 2 \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} + 1$$

The min occurs at $x^* = K^{-1} f = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$

$$= \frac{1}{11} \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{22} \\ \frac{5}{22} \end{pmatrix}$$

In Calc III, multi

$$\begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{pmatrix} = \begin{pmatrix} 8x - 2y + 3 \\ 6y - 2x - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$8x - 2y = -3$$
$$-2x + 6y = 2$$

$$\Longrightarrow \begin{pmatrix} 8 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \qquad !!$$

$$\frac{\partial}{\partial x}\left(4x^2 - 2xy + \underline{3y^2} + \underline{3}x \,\cancel{\boxed{-2}y} + \cancel{y}\right)$$

$$= 8x - 2y + 3$$

Don't do it this way!