Reminder HW 8 due Friday.

HW 9 not due next Friday

(in fact due 12/4)

Last time: minimization of quadratics

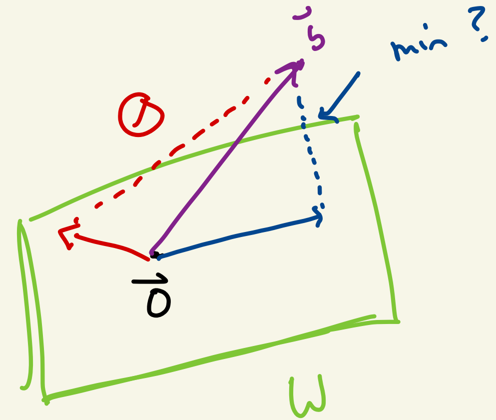
$$p(x) = \underbrace{x^T K x}_{\text{deg } 2} - \underbrace{2x^T f}_{\text{deg } 1} + \underbrace{c}_{\text{constant}}$$

If K is pos def, then $p(x)$ has a min!

It occurs at $\vec{x}^* = K^{-1} f$

the minimum value $p(x^*) = c - f^T x^*$.

Related Problem: Suppose $W \subseteq \mathbb{R}^n$ w/ dot product. W is a subspace. Let $\vec{b} \in \mathbb{R}^n$. What vector in W is closest to $\vec{b}$?



Find $\min \|\vec{w} - \vec{b}\|_{w \in W}$.

Turn this into a quadratic minimization!

$$\|w - b\|^2 = (w-b)\cdot(w-b)$$
$$= w \cdot w - 2w\cdot b + b \cdot b$$
$$= \|w\|^2 - 2w\cdot b + \|b\|^2$$

Suppose W has a basis $\vec{w}_1 \dots \vec{w}_k$.

Since basis vectors span

all vectors $\vec{w} \in W$ have the form

$$\vec{w} = x_1 \vec{w}_1 + x_2 \vec{w}_2 + \dots + x_k \vec{w}_k .$$

x_i coefficients

Idea: Find coefficients $x_1 \dots x_k$ which minimize $\|w-b\|$ and plug them into the linear comb.

$$\|w-b\|^2 = \|w\|^2 - 2w\cdot b + \|b\|^2$$

$$= \Big\| \sum_{i=1}^{k} x_i \vec{w}_i \Big\|^2 - 2\Big(\sum x_i \vec{w}_i\Big)\cdot b + \|b\|^2$$

bilinearity

$$= \Big(\sum x_i \vec{w}_i\Big)\cdot\Big(\sum x_i \vec{w}_i\Big) - 2\sum x_i (\vec{w}_i \cdot b) + \|b\|^2$$

$\downarrow$ FOIL
using bilinearity

$$= \sum_{i,j} (x_i \vec{w}_i \cdot x_j \vec{w}_j) - 2 \sum x_i (w_i \cdot b) + \|b\|^2$$

$$= \sum_{i,j} x_i x_j \underbrace{\boxed{(w_i \cdot w_j)}}_{x^T \boxed{K} x} \underbrace{\boxed{-\ 2}}_{\boxed{-\ 2}} \sum x_i \underbrace{\boxed{(w_i \cdot b)}}_{x^T \boxed{f}} + \underbrace{\boxed{\|b\|^2}}_{c}$$

$$= \underbrace{(x_1 \; \cdots \; x_k)}_{x^T} \underbrace{\begin{pmatrix} w_1 \cdot w_1 & w_1 \cdot w_2 & \\ w_1 \cdot w_2 & \ddots & \\ & & w_k \cdot w_k \end{pmatrix}}_{K} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}}_{x}$$

$$- 2 \underbrace{(x_1 \; \cdots \; x_k)}_{x^T} \underbrace{\boxed{\begin{pmatrix} w_1 \cdot b \\ w_2 \cdot b \\ \vdots \\ w_k \cdot b \end{pmatrix}}}_{f} + \underbrace{\boxed{\|b\|^2}}_{c}$$

$= x^T K x - 2x^T f + c$

$W = \mathrm{span}(w_1, \dots w_k)$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$ coordinates

$w = x_1 \vec{w}_1 + \dots + x_k \vec{w}_k$

$K = \begin{pmatrix} w_1 \cdot w_1 & & w_i \cdot w_j \\ & \ddots & \\ w_i \cdot w_j & & w_k \cdot w_k \end{pmatrix}$ = Gram matrix of $w_1, \dots w_k$. !!

$f = \begin{pmatrix} w_1 \cdot b \\ \vdots \\ w_k \cdot b \end{pmatrix}$

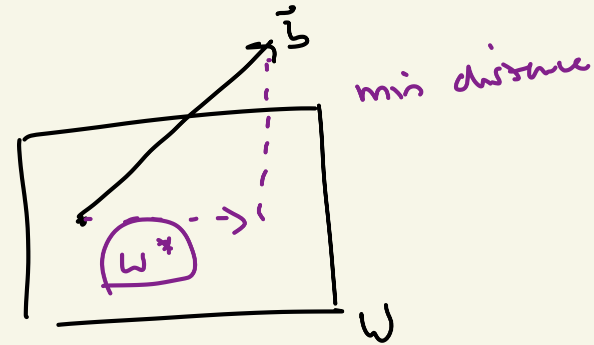
$c = \|b\|^2$

$\min_{w \in W} \|w - b\|$

$x^* =$ min coordinates $= K^{-1} f$

$\boxed{w^*} = x_1^* \vec{w}_1 + \dots + x_k^* \vec{w}_k = (\vec{w}_1 \dots \vec{w}_k) \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}^*$

$= \underline{(w_1 \cdots w_k)\, x^*}$ is the closest vector to b!



① Find basis of W, $w_1 \cdots w_k$

② Find Gram matrix K of your basis

③ $f = \begin{pmatrix} w_1 \cdot b \\ \vdots \\ w_k \cdot b \end{pmatrix}$

④ Let $A = (\vec{w}_1 \cdots \vec{w}_k)$.

Answer min distance $\|w - b\|$ occurs at $x^* = K^{-1} f$, vector

$w^* = Ax^*$, $d = \min \|w - b\| = \sqrt{c - f^T x^*}$

$= \sqrt{\|b\|^2 - f^T x^*}$

Slightly different formula.

$K =$ Gram-matrix of $w_1 \cdots w_k$ w/ dot product

$$= \begin{pmatrix} - w_1 - \\ \vdots \\ - w_k - \end{pmatrix} \begin{pmatrix} | & & | \\ w_1 & \cdots & w_k \\ | & & | \end{pmatrix} = A^T A$$

$$f = \begin{pmatrix} w_1 \cdot b \\ \vdots \\ w_k \cdot b \end{pmatrix} = \begin{pmatrix} - w_1 - \\ \vdots \\ - w_k - \end{pmatrix} \vec{b} = A^T b$$

$$x^* = K^{-1} f = (A^T A)^{-1} A^T b \qquad \text{where } A = (\vec{w}_1 \cdots \vec{w}_k)$$

$$w^* = A x^* = A(A^T A)^{-1} A^T b$$

optional

$$d = \min \| w - b \| = \sqrt{\|b\|^2 - \underline{x^*} \cdot \underline{f}} = \boxed{\sqrt{\|b\|^2 - \left((A^T A)^{-1} A^T b \right)^T (A^T b)}}$$

Process: Suppose $W \subseteq \mathbb{R}^n$ $\min_{w \in W} \|w - b\| = ?$, b given

① Find a basis $w_1 \ldots w_k$ of W

② $A = (w_1 \ldots w_k)$

③ $x^* = (A^T A)^{-1} A^T b$

④ $w^* = A x^*$ w^* is the vector of W closest to $\vec{b}$

⑤ $d^* = \sqrt{\|b\|^2 - x^* \cdot (A^T b)}$ Done. min distance d^*.

Ex Suppose $W = \text{span}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$. Let $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^*$.

- Find the minimum distance from W to b. (Asking for d^*)
- Which vector of W is closest to b. (Asking for w^*)

① $w_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad w_2 = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}.$

Span $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

Are $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$ independent?

② $A = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & -1 \end{pmatrix}$

Span $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}, \cancel{\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}}$

dependent!

③ $x^* = \underbrace{(A^TA)^{-1}}\underbrace{A^Tb}$

$$(A^TA)^{-1} \longrightarrow A^TA = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -3 \\ -3 & 14 \end{pmatrix}$$

$$(A^TA)^{-1} = \begin{pmatrix} 6 & -3 \\ -3 & 14 \end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} 14 & 3 \\ 3 & 6 \end{pmatrix}$$

$$= \frac{1}{75}\begin{pmatrix} 14 & 3 \\ 3 & 6 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{*} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = f$$

$$x^* = (A^T A)^{-1} A^T b = \frac{1}{75} \begin{pmatrix} 14 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{75} \begin{pmatrix} 20 \\ 15 \end{pmatrix} = \begin{pmatrix} 4/15 \\ 1/5 \end{pmatrix}$$

(4) $$w^* = \frac{4}{15} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = A x^* = \begin{pmatrix} 1 & 2 \\ 2 & -3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4/15 \\ 1/5 \end{pmatrix}$$

$$x_1^* u_1 + x_2^* u_2$$

$$= \begin{pmatrix} 2/3 \\ -1/15 \\ -7/15 \end{pmatrix}$$

vector in span $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

closest to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$!

(5) $$d = \sqrt{\|b\|^2 - x^{*T} f} = \sqrt{1 - (4/15, 1/5) \cdot (1, 2)}$$

$$= \sqrt{1 - \frac{4}{15} - \frac{6}{15}} = \sqrt{\frac{1}{3}}$$

<u>Thm</u> $\quad w^* = \text{proj}_W b$!

$$\text{proj}_W b = A(A^TA)^{-1}A^Tb$$

$P = A(A^TA)^{-1}A^T$ is called a projection matrix.