HW 7 on canvas!

Exam 2 next friday! (11/13)

Last time...

Alternate Gram-Schmidt algorithm

Input : $v_1 \dots v_n$ basis of V, $\langle -,- \rangle$

Output : $\underbrace{(u_1 \dots u_n)}_{\text{orthonormal basis}} \underbrace{\begin{pmatrix} r_{11} & r_{12} & & r_{1n} \\ & r_{22} & & \\ & & \ddots & \\ 0 & & & r_{nn} \end{pmatrix}}_{\text{upper triangular}} = (v_1 \dots v_n)$

$v_1 = r_{11} u_1$

$v_2 = r_{12} u_1 + r_{22} u_2$

$\vdots$

$v_n = r_{1n} u_1 + \dots + r_{nn} u_n$

- $r_{11} = \|v_1\|$
- $r_{12} = \langle v_2, u_1 \rangle$
- $r_{22} = \sqrt{\|v_2\|^2 - r_{12}^2}$
- $u_2 = \dfrac{v_2 - r_{12} u_1}{r_{22}}$

etc ...

???

upper Δ

backsubstitution!

$$\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \underbrace{\begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ 0 & & r_{nn} \end{pmatrix}}_{R}$$

This is called a QR factorization of the matrix

$(v_1 \cdots v_n) := A$.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} \\ \frac{-1}{\sqrt{3}} & \frac{5}{\sqrt{42}} & \frac{-1}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{-1}{\sqrt{3}} & -\sqrt{3} \\ 0 & \sqrt{\frac{14}{3}} & \sqrt{\frac{21}{2}} \\ 0 & 0 & \sqrt{\frac{7}{2}} \end{pmatrix}$$

What is the draw of Q?

Why is this matrix important?

It's columns form an orthonormal basis! This actually really helpful for solving many math problems.

Def: We say Q is an **orthogonal matrix** if its columns form a **orthonormal** basis of $\mathbb{R}^n$ w/r/t the dot product.

Ex: • $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ orthogonal since e_1, e_2, e_3 is orthonormal.

• $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} \\ \frac{-1}{\sqrt{3}} & \frac{5}{\sqrt{42}} & \frac{-1}{\sqrt{14}} \end{pmatrix}$ is orthogonal

• $Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal.

$-e_1, -e_2, e_3$ is still orthogonal!

- $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal.

e_2, e_1, e_3 is a orthonormal basis.

Note! Some orthogonal matrices are easy write down, but look this one. X.

$$\left(A \mid I \right) \xrightarrow[\text{reduce}]{\text{row}} \cancel{\left(I \mid A^{-1} \right)}$$

Properties

Prop) Alternate Def: A matrix Q is orthogonal $\boxed{\text{iff}}$ $Q^T = Q^{-1}$.

Pf: Let Q be orthogonal. $Q = (\vec{u}_1\ \vec{u}_2 \ldots \vec{u}_n)$.

Q^{-1} is the unique matrix s.t. $Q^{-1}Q = I$.

So to show that $Q^T = Q^{-1}$ all we have to do is show that $Q^T Q = I$.

(Q^T satisfies $XQ = I$ and inverse is unique so $Q^T = Q^{-1}$.)

$$Q^T Q = \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vdots \\ \vec{u}_n \end{pmatrix} \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{pmatrix}$$

def of matrix mult.

$$= \left(\vec{u}_i \cdot \vec{u}_j \right)_{ij}$$

But $\vec{u}_1 \ldots \vec{u}_n$ is an orthonormal basis!

- $\|u_i\|^2 = 1$ and
- $u_i \perp u_j$. $i \neq j$.

$\boxed{u_i \cdot u_i = 1}$ and $\boxed{u_i \cdot u_j = 0}$ $i \neq j$

So $Q^T Q = \begin{pmatrix} \boxed{u_1 \cdot u_1} & \boxed{u_1 \cdot u_2} & \cdots \\ \boxed{u_2 \cdot u_1} & \boxed{u_2 \cdot u_2} & \\ \vdots & & \ddots \\ & \cdots & \boxed{u_n \cdot u_n} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$

$= I.$

So $Q^T = Q^{-1}$.

($\Leftarrow$) If $Q^{-1} = Q^T \implies Q^T Q = I$

$\implies u_i \cdot u_i = 1$

$u_i \cdot u_j = 0$

$\implies \vec{u}_1 \dots \vec{u}_n$ orthonormal basis $\square$

$$\bullet \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} \\ \frac{-1}{\sqrt{3}} & \frac{5}{\sqrt{42}} & \frac{-1}{\sqrt{14}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{5}{\sqrt{42}} \\ \frac{2}{\sqrt{14}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{14}} \end{pmatrix}$$

<u>Prop</u>

- If Q is orthogonal matrix, then Q^T is also orthogonal.

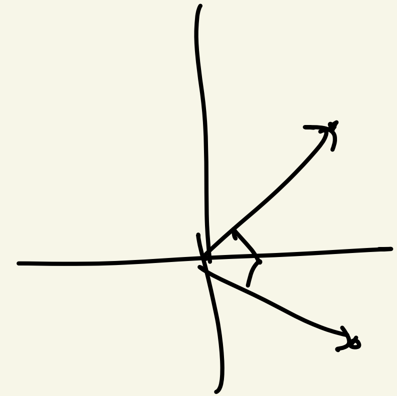
(if columns of Q are an orthonormal basis, then so do the rows of Q)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ || columns of Q^T

$$\left(\frac{1}{\sqrt{3}}, \frac{4}{\sqrt{42}}, \frac{2}{\sqrt{14}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{42}}, \frac{-3}{\sqrt{14}}\right)$$

$\left(\frac{-1}{\sqrt{3}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{14}}\right)$ form an orthonormal basis!

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$$\tilde{Q} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{Q}^T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

orthogonal basis aren't as nice!

· If Q is orthogonal, then Q^T is also orthogonal.

Pf: We know that $Q^T = Q^{-1}$ is a defining equation for being an orthogonal matrix.

To show that Q^T is orthogonal, we can show that

$$(Q^T)^T = (Q^T)^{-1}.$$

(Replace Q w/ Q^T in defining equation.)

$$\underline{(Q^T)^T} = Q^{TT} = Q = (Q^{-1})^{-1} = \underline{(Q^T)^{-1}}.$$

$Q^T = Q^{-1}$ since Q was orthogonal.

$\square$.

Prop Let P, Q be orthogonal matrices. Then PQ is also orthogonal.

Pf We need to show that $(PQ)^T = (PQ)^{-1}$.
(defining eq'n for PQ being orthogonal)

$$\underbrace{(PQ)^T} = Q^T P^T = Q^{-1} P^{-1} = \underbrace{(PQ)^{-1}}.$$

$\square$

$(\det A^T = \det A)$

Prop Let Q be an orthogonal matrix. Then $\det(Q) = \pm 1$.

Pf: $1 = \det I = \det(Q^{-1}Q) = \det Q^{-1} \det Q$

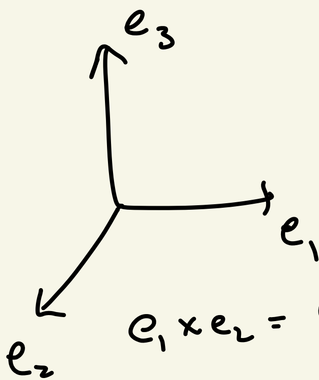
$= \det Q^T \det Q. = (\det Q)(\det Q)$

$$1 = (\det Q)^2 \rightsquigarrow \det Q = \pm 1. \qquad \square$$

<u>Ex</u> $\det \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 1 \qquad \det \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = -1$

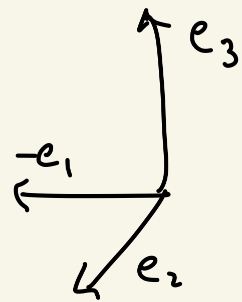
$\downarrow$

$-e_1, e_2, e_3$ forms a orthonormal basis.



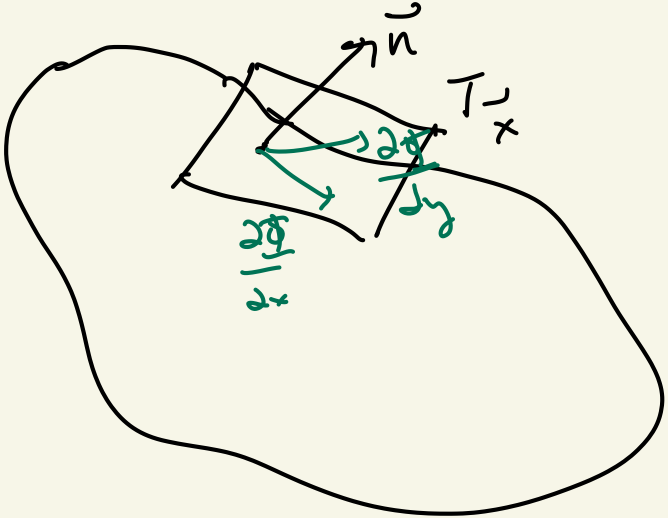
$e_1 \times e_2 = e_3$

" Right handed "



$-e_1 \times e_2 = -e_3 \neq e_3.$

" left handed "



Surface $\quad \int_S F(x,y)\cdot d\vec{s}$

$T_{\vec{x}}$ tangent space = $\{$all tangent vectors$\}$

= vector space of tangent vectors!

$\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}$ forms a basis of $T_{\vec{x}}$!

$\vec{n} = \frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y}$ needed to be outward!

$$\Updownarrow$$

$$\det\left(\frac{\partial \Phi}{\partial x} \quad \frac{\partial \Phi}{\partial y}\right) = 1 \neq -1.$$

$\det\left(\frac{\partial \Phi}{\partial x} \quad \frac{\partial \Phi}{\partial y}\right) = -1$ would be inward normal.

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$\det\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = 1$ but not an orthogonal matrix

$\det Q = \pm 1 \quad \not\Longrightarrow \quad Q$ is orthogonal.

✓ $\langle cv, w\rangle = c\langle v, w\rangle$

✓ $\langle v+u, w\rangle = \langle v, w\rangle + \langle u, w\rangle$

shorter eq'ns for linearity

✓ $\langle v, cw\rangle = c\langle v, w\rangle$

✓ $\langle v, w+u\rangle = \langle v, w\rangle + \langle v, u\rangle$

(redundant if $\langle -,- \rangle$ is symmetric)

Bilinearity = linearity + symmetry

$\langle v, w\rangle = v_1w_1 + v_1w_2 + v_2w_1 + v_2w_2$ <u>not</u> an inner product.

It is bilinear $\quad \langle v, w\rangle = (v_1\ v_2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

formulas like this are always bilinear

But this formula is not positive.

$$\langle (1,-1), (1,-1) \rangle = (1\ \ -1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= (1\ \ -1)\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

$\langle (1,-1), (1,-1) \rangle > 0$ is false!

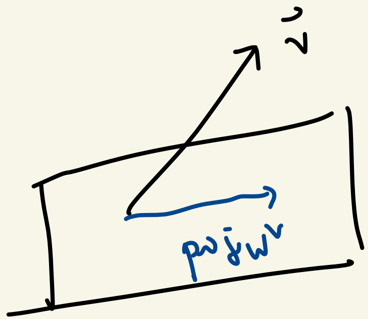
$$\langle cv, w \rangle = \langle c(v_1, v_2), (w_1, w_2) \rangle$$

$$= \langle (cv_1, cv_2), (w_1, w_2) \rangle$$

$$= (cv_1)w_1 + (cv_2)w_1 + (cv_1)w_2 + (cv_2)w_2$$

$$= c\left(v_1w_1 + v_2w_1 + v_1w_2 + v_2w_2 \right)$$

$= \quad c\langle v, w\rangle$



If $u_1, \dots, u_k$ is orthonormal basis W

then $\mathrm{proj}_W v = AA^T v$ ⟶ Gram matrix of rows

$A = \begin{pmatrix} u_1 & \dots & u_k \end{pmatrix}$ not square!