Ch. 7 Linear Transformations

Def: Let V, W be vector spaces. Let $T : V \longrightarrow W$ be a function

$$T : V \longrightarrow W$$

name of function

domain
- set of inputs

codomain
- set of possible outputs $\neq$ set of actual outputs

We say T is a linear transformation if for all vectors $v_1, v_2 \in V$ and scalars c

① $T(v_1 + v_2) = T(v_1) + T(v_2)$ and ② $T(cv_1) = cT(v_1)$.

We might call a linear transformation

- linear function
- linear operator

<u>Ex</u> $T: \mathbb{R}^2 \longrightarrow \mathbb{R}$

output

1 number

input a 2-vector $\begin{pmatrix} x \\ y \end{pmatrix}$

$T\begin{pmatrix} \boxed{x} \\ \boxed{y} \end{pmatrix} = \boxed{x} - \boxed{y}$ is a linear transformation.

Just like inner products or subspaces, we need to show that $T(x,y) = x - y$ satisfies the 2 properties!

$$\underline{T\left(\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}u\\v\end{pmatrix}\right)} = T\begin{pmatrix}\boxed{x+u}\\\boxed{y+v}\end{pmatrix} = \boxed{(x+u)} - \boxed{(y+v)}$$

$$= x-y+u-v = (x-y)+(u-v) = \underline{T\begin{pmatrix}x\\y\end{pmatrix}+T\begin{pmatrix}u\\v\end{pmatrix}}$$ ✓

① $T(\vec{v}_1+\vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

② $T(c\vec{v}_1) = cT(\vec{v}_1)$ $\qquad T:\mathbb{R}^2 \longrightarrow \mathbb{R}$

$$T(\boxed{x}\,\boxed{y}) = \boxed{x} - \boxed{y}$$

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = x-y+z$$

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}+\begin{pmatrix}u\\v\\w\end{pmatrix}\right)$$

$$T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}\boxed{cx}\\\boxed{cy}\end{pmatrix}$$

$$= \boxed{cx} - \boxed{cy} = c(x-y)$$

$$= cT\begin{pmatrix}x\\y\end{pmatrix}$$ ✓

□

- $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

In some sense we've studied this function

This is just matrix multiplication!

Ex

$$T : \underset{\substack{\text{Input} \\ \text{continuous functions} \\ \text{on } [a,b]}}{C^0[a,b]} \longrightarrow \underset{\substack{\text{Output} \\ \text{single number}}}{\mathbb{R}}$$

$$T(\overset{\text{input}}{f}) = \overbrace{\int_a^b f(x)\, dx}^{\text{single number}}$$

is linear!

① $T(f+g) = T(f) + T(g)$? ✓

② $T(cf) = c\,T(f)$? ✓

Calc I or II

① $$T(f+g) = \int_a^b f(x) + g(x)\,dx \overset{\checkmark}{=} \underbrace{\int_a^b f(x)\,dx}_{T(f)} + \underbrace{\int_a^b g(x)\,dx}_{T(g)}$$

$$= T(f) + T(g)$$

② $$T(cf) = \int_a^b c f(x)\,dx \underset{\substack{\uparrow \\ \text{calc I}}}{=} c \int_a^b f(x)\,dx = c\,T(f)$$

Aka integration is a

linear operator!

More examples

- $T : \underset{\substack{\uparrow \\ \text{Input} \\ \text{2-vector}}}{\mathbb{R}^2} \longrightarrow \underset{\substack{\uparrow \\ \text{Output} \\ \text{3-vector}}}{\mathbb{R}^3}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - 3y \\ x + y \\ 5x + 2y \end{pmatrix}$$ is linear!

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The reason this is linear is because $2x-3y$, $x+y$, $5x+2y$ are linear expressions (algebraically).

(No x^2, $\sin(x)$, x^3
nothing like that

- $T : \underset{\substack{\uparrow \\ \text{inputs} \\ \text{differentiable} \\ \text{functions}}}{C^1[a,b]} \longrightarrow \underset{\substack{\uparrow \\ \text{Output} \\ \text{continuous} \\ \text{functions}}}{C^0[a,b]}$

$|x| \in C^0[-1,1]$

$|x| \notin C^1[-1,1]$

$\frac{d}{dx}|x|$ not defined
no deriv at $x=0$!

$T(f) = \frac{df}{dx}$ is a linear operator or a linear transformation!

Non examples

$T : \mathbb{R}^2 \to \mathbb{R}$ $\qquad T\begin{pmatrix}x\\y\end{pmatrix} = x + y \underset{\substack{\uparrow \\ \text{culprit}}}{-2}$ $\qquad$ not linear!

$T\begin{pmatrix}x\\y\end{pmatrix} = x^2 + y^2 - 3$ $\qquad$ not linear!

$$T\left(c\begin{pmatrix}x\\y\end{pmatrix}\right) = T\begin{pmatrix}cx\\cy\end{pmatrix} = (cx)^2 + (cy)^2 - 3$$

$$= c^2x^2 + c^2y^2 - 3$$

$$= \underline{c}^2(\underline{x}^2 + y^2) - \underline{3} \neq cT\begin{pmatrix}x\\y\end{pmatrix}$$

$$cT\begin{pmatrix}x\\y\end{pmatrix} = c(x^2+y^2-3) = \underline{c}x^2 + cy^2 - 3\underline{c}$$

✗ not linear

Why?

Suppose $T: V \longrightarrow W$ and $S: V \longrightarrow W$ are two linear functions. Let $\vec{v}_1 \dots \vec{v}_n$ be a basis of V.

Then if $T(\vec{v}_i) = S(\vec{v}_i)$ for all basis vectors $\vec{v}_i$,

then $T = S$.

- If S, T agree on the basis, they agree everywhere. *
- Bases determine the values of a linear function. *

Come back to this

Pf Let $\vec{v}$ be any vector in V.

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n .$$

$$
\begin{aligned}
\boxed{S(\vec{v})} &= S(a_1\vec{v}_1 + \dots + a_n\vec{v}_n)\\
&\overset{\textcircled{1}}{=} S(a_1\vec{v}_1) + S(a_2\vec{v}_2) + \dots + S(a_n\vec{v}_n)\\
&\overset{\textcircled{2}}{=} \underset{\downarrow}{a_1 S(\vec{v}_1)} + \underset{\downarrow}{a_2 S(\vec{v}_2)} + \dots + \underset{\downarrow}{a_n S(\vec{v}_n)}\\
&\overset{\text{assumption}}{=} a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \dots + a_n T(\vec{v}_n)\\
&= T(a_1\vec{v}_1 + \dots + a_n\vec{v}_n) = \boxed{T(\vec{v})}. \qquad \square
\end{aligned}
$$

Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be any linear function.

Then there exists a $m \times n$ matrix A

such that $T(\vec{x}) = \boxed{A}\,\vec{x}$.

- All linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$ are matrix multiplication!

Pf Consider the standard basis on $\mathbb{R}^n$ $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.

output $\nearrow$ $T(\vec{e}_i) \in \mathbb{R}^m$ since $T: \mathbb{R}^n \to \mathbb{R}^m$.

$$\text{Define } \boxed{A} = \begin{pmatrix} | & | & & | \\ T(\vec{e}_1) & T(e_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{pmatrix}.$$

m rows n columns $\underline{m \times n}$

So why $T(\vec{x}) = A\vec{x}$?

$c_1 v_1 + \dots c_n v_n$

$= \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$\boxed{T(\vec{x})} = T(x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n)$$

$$\left.\begin{aligned} &= T(x_1\vec{e}_1) + \dots + T(x_n\vec{e}_n) \\ &= \underbrace{x_1}\,\underbrace{T(\vec{e}_1)} + \dots + \underbrace{x_n}\,\underbrace{T(\vec{e}_n)}_{\text{vectors}} \end{aligned}\right\} \text{T is linear!}$$

$$= \begin{pmatrix} T(e_1) & \dots & T(\vec{e}_n) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\downarrow$$

$$= A \qquad \vec{x} \qquad = \boxed{A\vec{x}} \quad \square$$

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x-3y\\x+y\\5x+2y\end{pmatrix} \; *$$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\cdot 1 - 3\cdot 0\\1+0\\5\cdot 1 + 2\cdot 0\end{pmatrix} = \begin{pmatrix}2\\1\\5\end{pmatrix}$$

$$T\begin{pmatrix}x\\y\end{pmatrix} = \underline{\begin{pmatrix}2 & -3\\1 & 1\\5 & 2\end{pmatrix}}\begin{pmatrix}x\\y\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$$

$$A = \left(T(\vec{e}_1) \cdot T(e_2)\right) = \begin{pmatrix}2 & -3\\1 & 1\\5 & 2\end{pmatrix}$$

Practically!
look at coefficients

Proof: We don't know ahead of time

$$T(\vec{x}) = \begin{pmatrix}a_{11}x_1 + \dots + a_{1n}x_n\\ a_{21}x_1 + \dots + a_{2n}x_n\\ \vdots\end{pmatrix} \qquad A = \begin{pmatrix}a_{11} \cdots a_{1n}\\ a_{21} \cdots a_{2n}\\ \vdots\end{pmatrix}$$