Reminder: No homework due 11/27

But hw 9 due 12/4.

Last time: Linear transformations! $T: V \longrightarrow W$

- $T(v_1 + v_2) = T(v_1) + T(v_2)$
- $T(cv_1) = cT(v_1)$

Proved that if $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

then $T(\vec{x}) = A\vec{x}$ where $m \times n$.

(nothing new if V, W are $\mathbb{R}^n, \mathbb{R}^m$)

$\frac{d}{dx}: C^1[a,b] \longrightarrow C^0[a,b]$ is a linear transformation

$\int_a^b - \, dx : C^0[a,b] \longrightarrow \mathbb{R}^1$ is a linear transformation

$\frac{d^2}{dx^2} + \frac{d}{dx}$ is also linear!

$C^2[a,b] \longrightarrow C^0[a,b]$

twice differentiable!

Today: <u>Thm</u> If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then there exists a unique A, $m \times n$ such that

$T(\vec{x}) = A\vec{x}$.

A can change depending on chosen basis of $\mathbb{R}^n$, $\mathbb{R}^m$!!

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x - 1y \\ 2x - 3y \end{pmatrix}$

Last time: $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

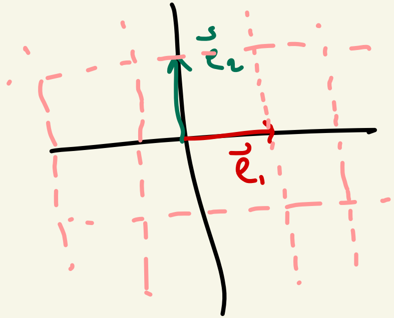
$$T\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad T\boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$$

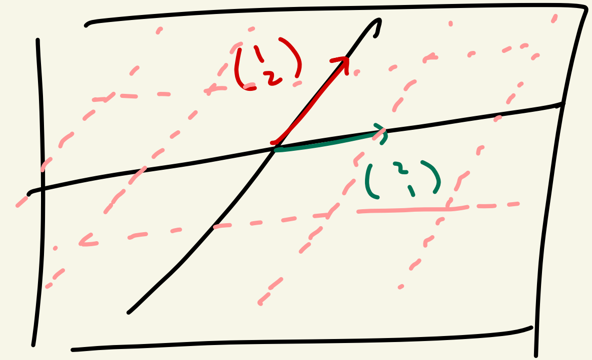
Message: All along this calculation was in the standard basis.

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

There's more than one basis.



$\mathbb{R}^2$



$\mathbb{R}^2$

This is a perfectly good way to draw $\mathbb{R}^2$.

Q: Given a transformation like
$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & -1\\2 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix},$$
how can I represent this in the $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$ basis? (Ex)

Def: Let $\vec{v} \in \mathbb{R}^n$. Suppose $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of $\mathbb{R}^n$. Then the <u>coordinates</u> of $\vec{v}$ in terms of $\vec{v}_1 \ldots \vec{v}_n$ are the coefficients $c_1 \ldots c_n$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n.$$

The vector $\vec{v}$ written in $\vec{v}_1 \ldots \vec{v}_n$ <u>coordinates</u> is the column vector $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$.

Ex: In the standard basis

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3e_1 + 2e_1 = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$(3,2)$ would be the <u>coordinates</u>.

what about $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$?

$$\begin{pmatrix}3\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix} + \frac{4}{3}\begin{pmatrix}2\\1\end{pmatrix}.$$

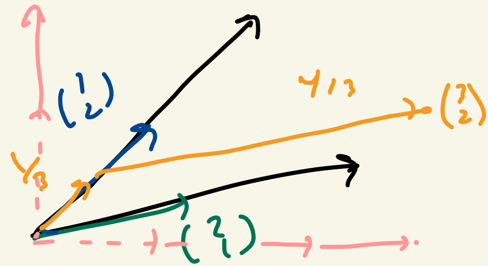
$\begin{pmatrix}3\\2\end{pmatrix}$ "=" $\begin{pmatrix}1/3\\4/3\end{pmatrix}$ in $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$ **coordinates**!

$$\begin{pmatrix}3\\2\end{pmatrix} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix}$$

Solve for $c_1 c_2$

$$\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}^{-1}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1/3\\4/3\end{pmatrix}$$



Q: Given a vector $\vec{v}$ and a basis $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$.

what are the coordinates of $\vec{v}$ for β?

(Above $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$
$\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$)

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

$$\vec{v} = \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{pmatrix}^{-1} \vec{v}$$

This is the formula for the coordinates $c_1 \dots c_n$.

We say $\vec{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_\beta$ ← this is $\vec{v}$ represented in β coordinates

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix}_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}^{-1}}^{S^{-1}} \boxed{\vec{v}}$$

β coordinates

$$v \in \mathbb{R}^n \qquad v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\vec{e}}$$ standard coordinates

$$\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}^{-1} : \underline{\mathbb{R}^n_{\vec{e}}} \longrightarrow \underline{\mathbb{R}^n_\beta} \qquad \beta = \{\vec{v}_1, \dots, \vec{v}_n\}$$

$$\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}^{-1} = S^{-1}$$

Given $S^{-1} = (\vec{v}_1 \dots \vec{v}_n)^{-1}$ transforms standard coord. into

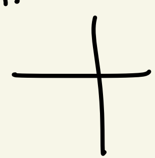
β - coord., how can we go back? It's S!

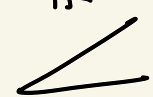
$$\begin{tikzcd} \mathbb{R}^n_e \arrow[r, bend left, "S^{-1}"] & \mathbb{R}^n_\beta \arrow[l, bend left, "S"] \end{tikzcd}$$

$$\vec{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}_\beta = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

__Ex__

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, bend left, "\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1}"] & \mathbb{R}^2 \arrow[l, bend left, "\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}"] \end{tikzcd}$$





$\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$$

$$\frac{-1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

Q: Given a transformation $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & -1\\ 2 & -3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ in standard coordinates, how we represent it in $\beta = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$ coordinates?

$$\begin{pmatrix}1\\0\end{pmatrix} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix}1\\0\end{pmatrix}$$

$B\vec{v}_\beta = T(v_\beta)$ in particular.

$$B \neq \begin{pmatrix}1 & -1\\ 2 & -3\end{pmatrix} \;!!$$

$$T\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1 & -1\\ 2 & -3\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$ in standard coord.

$$T\begin{pmatrix}1/3\\4/3\end{pmatrix} \neq \begin{pmatrix}1 & -1\\ 2 & -3\end{pmatrix}\begin{pmatrix}1/3\\4/3\end{pmatrix} = \begin{pmatrix}-1/3\\2/3\end{pmatrix}$$

$$\underline{\underline{\begin{pmatrix}1 & -1\\ 2 & -3\end{pmatrix}}}\begin{pmatrix}1/3\\4/3\end{pmatrix} \neq \begin{pmatrix}-1/3\\2/3\end{pmatrix}$$

this does not respect β-coordinates!

<u>In general</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ represented by A in st. coord.

$$\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$$

$$\begin{tikzcd}
\mathbb{R}^n_\beta \arrow[r, "B"] \arrow[d, shift right, "S"'] & \mathbb{R}^n_\beta \arrow[d, shift right, "S"'] \\
\mathbb{R}^n_e \arrow[r, "A"'] \arrow[u, shift right, "S^{-1}"'] & \mathbb{R}^n_e \arrow[u, shift right, "S^{-1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{R}^n_e \arrow[r, bend left, "S^{-1}"] & \mathbb{R}^n_\beta \arrow[l, bend left, "S"]
\end{tikzcd}$$

$$S = (\vec{v}_1 \dots \vec{v}_n)$$

Functions are arrows!

Matrix multiplication is arrows.

What's B?

We can go the long way around!

$B = S^{-1} A S$ this is the change of basis formula.

B is the transformation $T(\vec{x}) = A\vec{x}$ written in β - coord.

Q. Write the transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ in $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ coordinates.

A: Change of basis says $B = S^{-1} A S$

where $A = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 1 \\ -4 & 1 \end{pmatrix}$$

$$= \frac{-1}{3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -7/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$$

In standard coord. $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{T}$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ coord. $T\begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$ $\quad \begin{pmatrix} -7/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$

$\begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$ and $\begin{pmatrix} -7/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$ represent the same transformation

in different bases!

$B \sim A$ if $\exists S$ s.t. $B = S^{-1}AS$

$$\det(B) = \det(S^{-1}AS) = \det\cancel{(S^{-1})}\det(A)\det\cancel{(S)}$$
$$= \det(A).$$

$P = \begin{pmatrix} \text{proj}_W e_1 & \text{proj}_W e_2 & \text{proj}_W e_3 \end{pmatrix}$

Pick an orthonormal basis $\vec{u}_1, \vec{u}_2$ of W

$P = \begin{pmatrix} (e_1 \cdot u_1)u_1 + (e_1 \cdot u_2)u_2 & (e_2 \cdot u_1)u_1 + (e_2 \cdot u_2)u_2 & (e_3 \cdot u_1)u_1 + (e_3 \cdot u_2)u_2 \end{pmatrix}$

Simplify $= \begin{pmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \end{pmatrix} \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} = Q^T Q$

$P^2 = (Q^T Q)(Q^T Q) = Q^T \cancel{(Q Q^T)} Q$

$= Q^T Q = P$