Last time ...

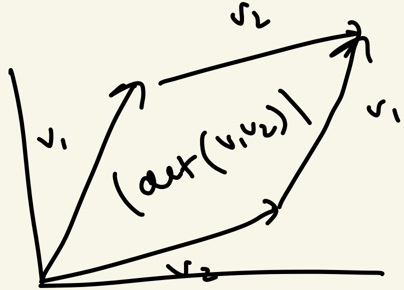
Def A matrix Q is orthogonal if the columns form an orthonormal basis of $\mathbb{R}^n$ w.r.t. dot product.

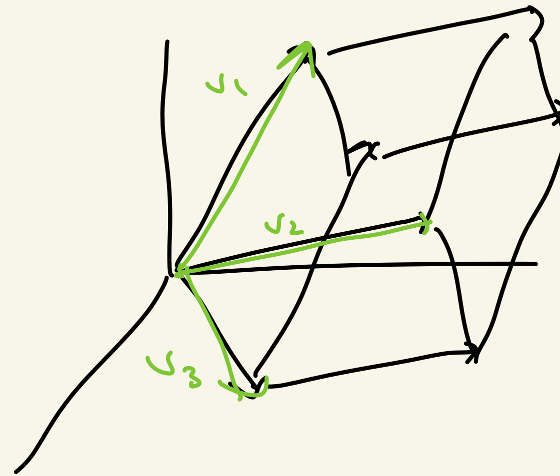
Properties: Q is orthogonal iff $Q^{-1} = Q^T$. $(Q^T Q = I)$

- Q^T also orthogonal (rows of Q form orthonormal basis)
- PQ orthogonal
- $\det Q = \pm 1$

$|\det(v_1 \dots v_n)|$ $n \times n$ matrix

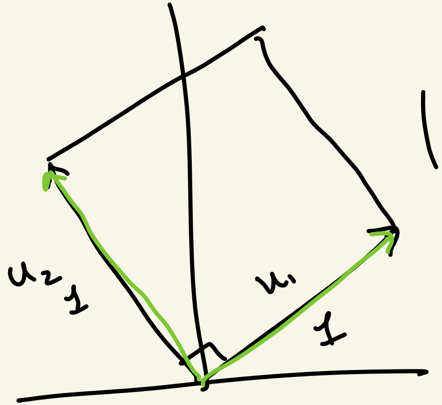
$=$ volume of the parallelogram formed by $v_1 \dots v_k$





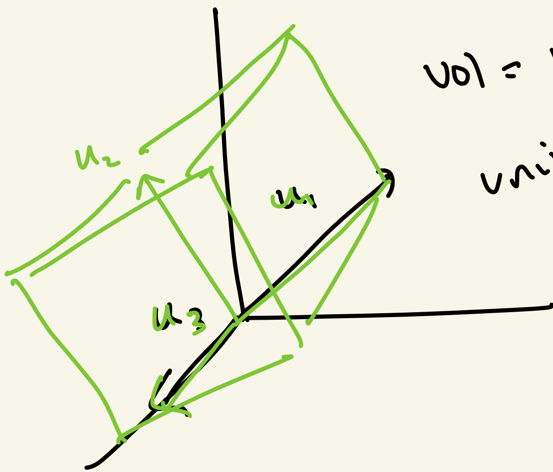
$V = |\det(v_1 v_2 v_3)|$

$\det Q = \pm 1$



$$|\det(u_1, u_2)| = 1$$

unit square



$$\text{vol} = 1 = |\det(u_1\, u_2\, u_3)|$$

unit cube

Prop Let Q be a 2×2 orthogonal matrix. Then

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$(\det = 1)$ $\qquad\qquad$ $(\det = -1)$

rotation matrix $\qquad\qquad$ rotation + reflection

Pf Suppose $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\begin{pmatrix} a \\ c \end{pmatrix}$ unit

$\begin{pmatrix} b \\ d \end{pmatrix}$ unit $\qquad \begin{pmatrix} a \\ c \end{pmatrix} \perp \begin{pmatrix} b \\ d \end{pmatrix}$

$$\Longrightarrow Q^T Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a^2 + c^2 = 1 \qquad ab + cd = 0$

$b^2 + d^2 = 1$

same!

$$\begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\cos\theta = \frac{v \cdot w}{\|v\| \cdot \|w\|}$$

$a^2+c^2 = 1$ (a,c) is on the unit circle

$\boxed{(\cos\theta, \sin\theta)}$ is a unit vector

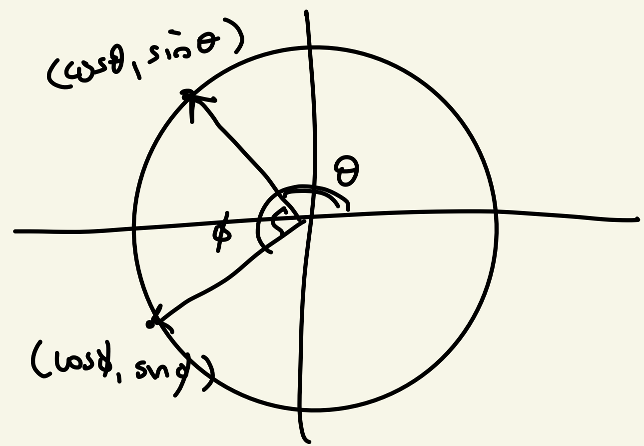
$b^2+d^2 = 1$ (b,d) also.

$$\left(\frac{v \cdot w}{\|v\|\|w\|} \right) \_\_\_ \Big)$$

$$\begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} b \\ d \end{pmatrix} = ab+cd = 0$$

Let $(a,c) = (\cos\theta, \sin\theta)$

$(b,d) = (\cos\phi, \sin\phi)$.



$$ab + cd = 0$$

$$\cos\theta\cos\phi + \sin\theta\sin\phi = 0$$

$$\cos(\theta - \phi) = 0$$

$$\Longrightarrow \quad \theta - \phi = \pm \frac{\pi}{2} \quad \Longleftrightarrow \quad \phi = \theta \pm \frac{\pi}{2}$$

If $\phi = \theta + \frac{\pi}{2}$

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \checkmark$$

If $\phi = \theta - \frac{\pi}{2}$

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \cos(\theta - \pi/2) \\ \sin(\theta - \pi/2) \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \quad \checkmark$$

$\square$

Remember Alternative G-S.

$$\begin{pmatrix} w_1 \cdots w_n \end{pmatrix} = \begin{pmatrix} u_1 \cdots u_n \end{pmatrix} \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ 0 & \ddots & \vdots \\ & & r_{nn} \end{pmatrix}$$

$$A \quad = \quad Q \quad\quad R$$

orthonormal basis as columns

upper Δ

orthogonal matrix!

Thm Let A be a $\boxed{\text{invertible}}$ matrix ? Then A has a decomposition, $\underline{A = QR}$ where Q is orthogonal and R is upper triangular. This decomposition is unique up to positivity of the diagonal entries of R.

alt G-S

Pf: let A be an invertible matrix. Remember from the Fundamental thm that A^{-1} exists iff the columns of A form a basis!

let $A = (w_1 \dots w_n)$ where $w_1 \dots w_n$ are a basis

By Alt. G-S $A = (w_1 \dots w_n) = (u_1 \dots u_n) \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ 0 & & r_{nn} \end{pmatrix}$

$= QR.$

Where in the A-G-S algorithm did we have to make choices?

Why is $A = QR$ almost unique?

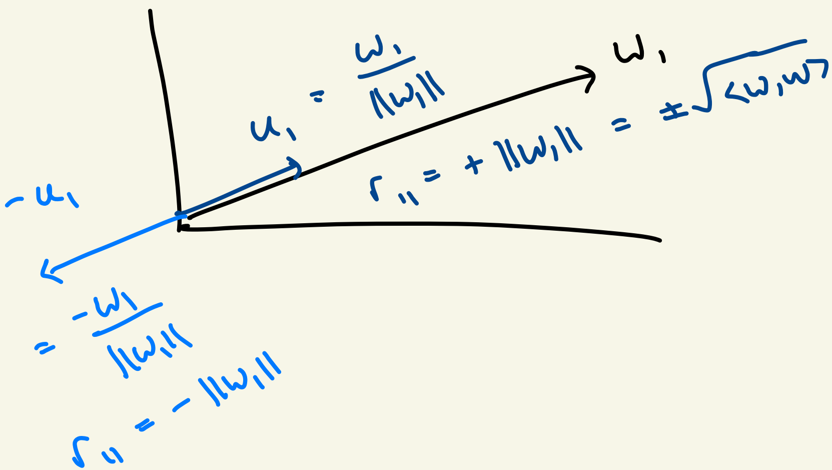
$\vec{w}_1 = r_{11} \vec{u}_1 \rightsquigarrow r_{11} = \pm \|w_1\| \quad \frac{w_1}{\|w_1\|}, \frac{-w_1}{\|w_1\|}$

So in the algorithm we pick $+\|w_1\|$.

$$r_{ii} = \pm\sqrt{\|w_i\|^2 - r_{1i}^2 - \cdots - r_{i,i-1}^2}$$

In the algorithm we picked $+\sqrt{}$.

So that's why $A = QR$ is unique up to choice of r_{ii}, the diagonal entries of R.



G-S

$w_1 \dots w_n \longrightarrow v_1 \dots v_n$ orthonormal

$v_1 = w_1$

$v_2 = w_2 - \dfrac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

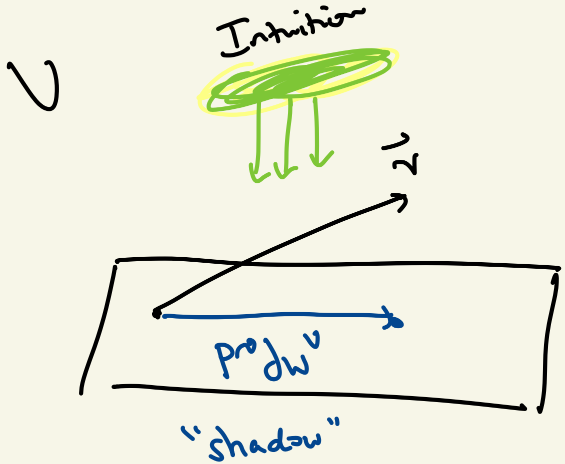
Alt G-S

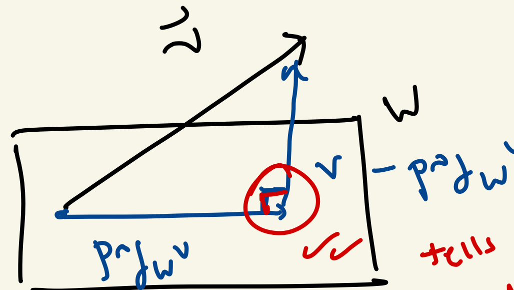
$w_1 \dots w_n \longrightarrow u_1 \dots u_n$ orthonormal

$r_{11} = \|w_1\| \qquad u_1 = \dfrac{w_1}{\|w_1\|}$

$r_{12} =$ etc.

§ 4.4 Projection onto a subspace.





tells how it works mathematically

<u>Def</u> Let $v \in V$ and $W \subseteq V$ a subspace.

Define $\text{proj}_W \vec{v}$ as the unique vector such that

IOU

$$\text{proj}_W v \in W \quad \text{and} \quad v - \text{proj}_W v \perp \text{proj}_W v.$$

Prop Suppose W has a finite orthonormal basis $u_1 \dots u_k$. (not a basis of V)

then $\text{proj}_W v = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k$ where $c_i = \langle v, u_i \rangle$.

If ~~$u_1 \dots u_k$~~ $v_1 \dots v_k$ an orthogonal basis,

then $\text{proj}_W v = a_1\vec{v}_1 + \dots + a_k\vec{v}_k$

where $a_i = \dfrac{\langle v, v_i \rangle}{\|v_i\|^2}$.

(4.1.21
4.1.23)

then $\text{proj}_W v = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k$ where $c_i = \langle v, u_i \rangle$.

Pf: By def $\text{proj}_W v \in W$ and $v - \text{proj}_W v \perp \text{proj}_W v$.

We need to show that our formula for $\text{proj}_W v$ satisfies these properties.

- $\text{proj}_W v = c_1\vec{u}_1 + \dots + c_k\vec{u}_k$ so $\in W$. (remember subspace properties $u_1 \dots u_k$ basis of W)

- $\langle v - \text{proj}_W v, \text{proj}_W v \rangle = \langle v, \text{proj}_W v \rangle - \langle \text{proj}_W v, \text{proj}_W v \rangle$

$= \langle v, c_1\vec{u}_1 + \dots c_k u_k \rangle - \langle c_1 u_1 + \dots + c_k u_k, c_1\vec{u}_1 + \dots + c_k\vec{u}_k \rangle$

$c_1 = \langle v, u_1 \rangle$
by def

$$= \underline{c_1} \underline{\langle v, u_1 \rangle} + \dots + c_k \langle v, u_k \rangle - c_1^2 - c_2^2 - \dots - c_k^2$$

$$= c_1^2 + c_2^2 + \dots + c_k^2 - c_1^2 - \dots - c_k^2 = 0 \qquad \square$$

Reminder OH : Th 12-1

HW7 : Friday

Exam 2 : Next Friday 11/13