HW7 : due tonight!

Exam 2 : next Friday (same policies as last time!)

Review materials post today

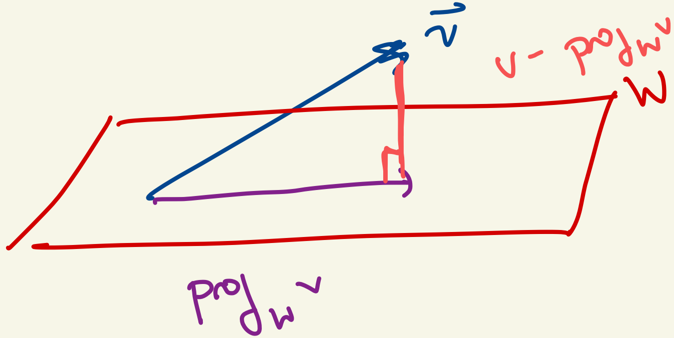
Last time:

Projection onto subspace

Theory: Given a vector $\vec{v}$, and a subspace W,

$\mathrm{proj}_W \vec{v}$ is the unique vector s.t. $\mathrm{proj}_W v \in W$

and $v - \mathrm{proj}_W v \perp \mathrm{proj}_W v$.



Given a <u>orthogonal</u> basis $v_1 \dots v_k$ of W

$$\text{proj}_W v = a_1 v_1 + \dots + a_k v_k \qquad (\text{if } u_1 \dots u_k$$

$$a_i = \frac{\langle v, v_i \rangle}{\|v_i\|^2} \qquad \|u_i\|^2 = 1 \,)$$

<u>Ex</u>: $\vec{v} = (1,0,0) \quad W = \text{span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$

Compute $\text{proj}_W v$.

So is $\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$ an orthogonal basis of W? ✓

$$\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix}^T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1 & -2 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = 1-2+1 = 0$$

$$\underline{\text{proj}_W v} = \frac{\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}}{\begin{pmatrix}1 & -2 & 1\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}}\begin{pmatrix}1\\-2\\1\end{pmatrix} + \frac{\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}{\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\frac{\vec{v}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1$$

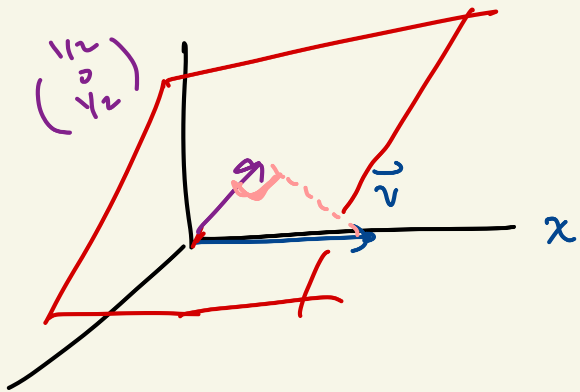
$$= \frac{1}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \underline{\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}} \boxed{\in W}$$

$$\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} \in W = \text{span}\left(\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$$

Should have $\boxed{v - \text{proj}_W v \perp \text{proj}_W v.}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -1/2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ -1/2 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{4} + 0 - \frac{1}{4} = 0 \quad \checkmark$$



G-S revisited

$w_1 \dots w_n$ basis $\longrightarrow$ orthogonal basis $v_1 \dots v_n$

the v's are orthogonal to each other, but not w's.

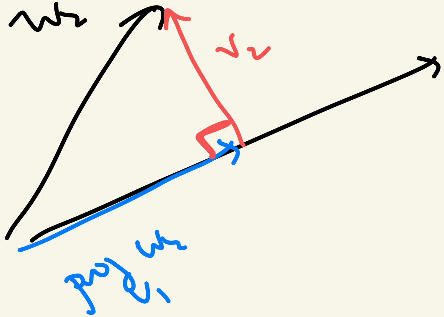
$$v_1 = w_1$$

$$v_2 = w_2 - \boxed{\frac{\langle w_2, v_1\rangle}{\|v_1\|^2}} v_1 \rightsquigarrow \operatorname{proj}_{\operatorname{span}(v_1)} w_2$$

$$v_2 = w_2 - \operatorname{proj}_{v_1} w_2 \perp v_1$$

v_1 is an orthogonal basis for $\operatorname{span}(v_1)$

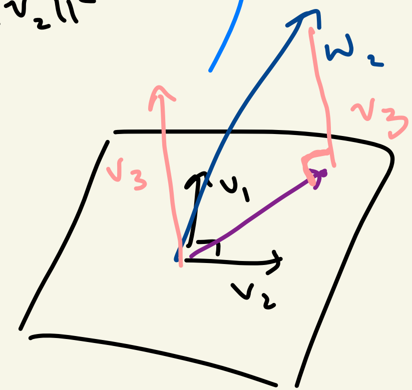
$v_2 \perp v_1$ 2^{nd} step is really projecting w_2 onto w_1 and taking orthogonal complement



$w_1 = v_1$ $\qquad$ $v_1 \perp v_2$.

$$v_3 = w_3 - \left(\frac{\langle w_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} v_2 \right)$$

$$= w_3 - \mathrm{proj}_{\mathrm{span}(v_1, v_2)} w_3$$



Orthogonal subspaces / orthogonal complements

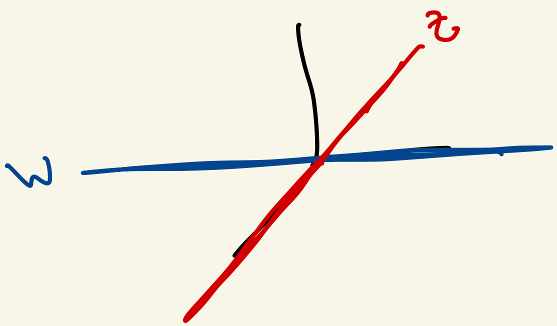
Def Let V be an inner product space. $W, Z \subseteq V$ subspaces.

We say W is orthogonal to Z $(W \perp Z)$

if $\forall \vec{w} \in W$, $\vec{z} \in Z$, $\langle \vec{w}, \vec{z} \rangle = 0$.

(for all)

Ex $V = \mathbb{R}^3$ $W = \text{span}(e_1)$ $Z = \text{span}(e_2)$



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$w = (w_1, 0, 0)$, $z = (0, z_2, 0)$

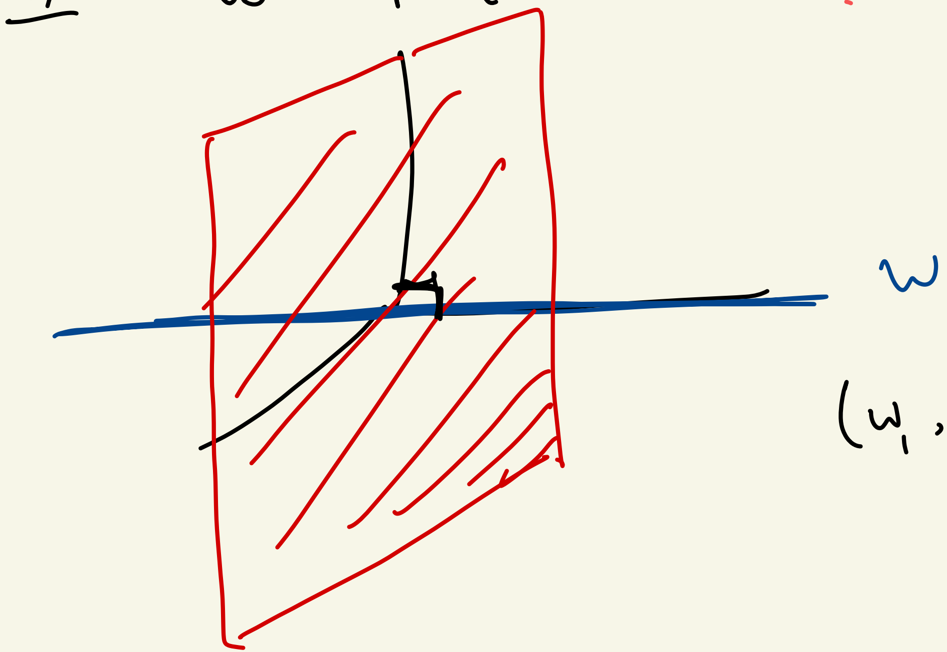
$w \cdot z = 0$!

So $W \perp Z$.

(Z could be bigger)

Ex'

$W = \mathrm{span}(e_1)$ $\qquad Z = \mathrm{span}(e_2, e_3)$



$W \perp Z$

these are orthogonal subspaces.

$(w_1, 0, 0) \perp (0, z_2, z_3).$

Biggest Z could be! there are no other vectors $\perp W$.

Def let $W \subseteq V$ of an inner product space.

Define $W^{\perp}$ (called "W-perp") to be

$$W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$$

= all vectors orthogonal to every vector in W.

Ex If $W = \text{span}(e_1) \subseteq \mathbb{R}^3$

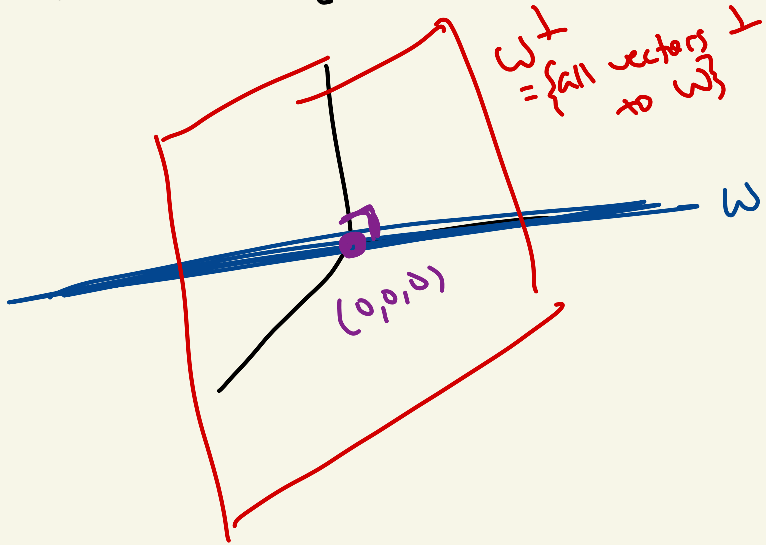
then $W^{\perp} = \{ (x,y,z) \in \mathbb{R}^3 \mid (x,y,z) \cdot \vec{w} = 0 \ \forall w \in W \}$

$$= \{ (x,y,z) \mid (x,y,z) \cdot (w_1, 0, 0) = 0 \}$$

$$= \{ (x,y,z) \mid x w_1 = 0 \ \forall w_1 \in \mathbb{R} \} = \{ (x,y,z) \mid x = 0 \}$$

$$W^\perp = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\} = \{(0,y,z)\}$$

$$= \operatorname{span}(e_2, e_3)$$



Prop $W^\perp$ is a subspace!

Pf ① Claim: $\vec{0} \in W^\perp$. Indeed $\langle 0, w\rangle = 0$ all the time!

$\Rightarrow 0 \in W^\perp$.

② Let $\vec{z}_1, \vec{z}_2 \in W^\perp$. $z_1 \in W^\perp \Rightarrow \langle z_1, w \rangle = 0$ same for z_2.

$\vec{z}_1 + \vec{z}_2 \overset{?}{\in} W^\perp$?

by assumption

Let $w \in W$, $\langle \vec{w}, \vec{z}_1 + \vec{z}_2 \rangle = \underbrace{\langle \vec{w}, \vec{z}_1 \rangle}_{0} + \underbrace{\langle \vec{w}, \vec{z}_2 \rangle}_{0}$

$= 0 + 0 = 0$

$\Rightarrow \vec{z}_1 + \vec{z}_2 \perp w \Rightarrow \vec{z}_1 + \vec{z}_2 \in W^\perp$ ✓

③ Let $c \in \mathbb{R}$, $\vec{z}_1 \in W^\perp$. Why is $c\vec{z}_1 \in W^\perp$?

$\underline{\langle c\vec{z}_1, w \rangle} = c\underbrace{\langle \vec{z}_1, \vec{w} \rangle}_{0} = c \cdot 0 = 0 \quad \forall w \in W.$

$c\vec{z}_1 \in W^\perp$ $\square$

Prop $W \cap W^{\perp} = \{\vec{0}\}$ for all subspaces $W \subseteq V$.

Pf Suppose $\vec{v} \in W \cap W^{\perp}$, so $v \in W$ and $\vec{v} \in W^{\perp}$.

But if $\vec{v} \in W^{\perp} \Rightarrow \langle v, w \rangle = 0 \quad \forall w \in W$.

In particular $\vec{v} \in W \Rightarrow \langle v, v \rangle = 0$

$$\Rightarrow \|v\|^2 = 0 \xRightarrow[\text{positivity}]{} \vec{v} = 0.$$

$$\Rightarrow W \cap W^{\perp} = \{\vec{0}\}. \qquad \square$$

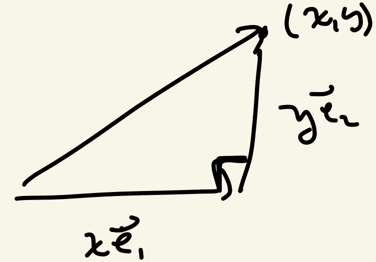
Thm Let $W \subseteq V$ and further assume that $\dim W < \infty$.

Then $\forall v \in V$ can be decomposed

$$v = \underbrace{w}_{\mathrm{proj}_W v} + \underbrace{z}_{v - \mathrm{proj}_W v}$$

where $w \in W$ and $z \in W^{\perp}$. Furthermore this decomposition is unique.

$(\langle z, w \rangle = 0)$



Pf Well $\dim W = k \Rightarrow w_1, \dots w_k$ basis.

$\xRightarrow{G-S}$ $v_1, \dots v_k$ orthogonal basis of W.

Let $\vec{w} = \mathrm{proj}_W \vec{v} = a_1 v_1 + \dots + a_k v_k$.

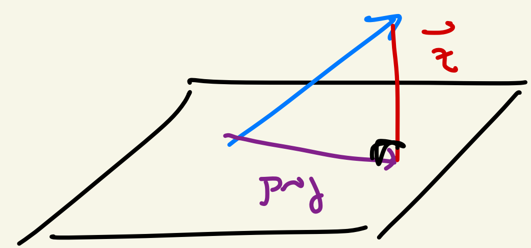
$a = \dfrac{\langle w, v_i \rangle}{\|v_i\|^2}$

Let $\vec{z} = v - \text{proj}_W v$. We claim that this the decomposition.

First, $w + z = \cancel{\text{proj}_W v} + (v - \cancel{\text{proj}_W v}) = v$ ✓

Second, $\vec{w} \in W$ because $\text{proj}_W v \in W$.

$\vec{z} \in W^\perp$ since $v - \text{proj}_W v \perp \text{proj}_W v$



z actually $\perp \vec{w} \;\; \forall \vec{w} \in W$.

--- IOU.

$$Q = \begin{pmatrix} q_{11} & & 0 \\ & \ddots & \\ 0 & & q_{nn} \end{pmatrix} \qquad Q^T = Q^{-1}$$

$$\begin{pmatrix} q_{11} & & \\ & \ddots & \\ & & q_{nn} \end{pmatrix}^T = \begin{pmatrix} q_{11} & & \\ & \ddots & \\ & & q_{nn} \end{pmatrix}^{-1}$$

simplify it to solve for q_{ii}.