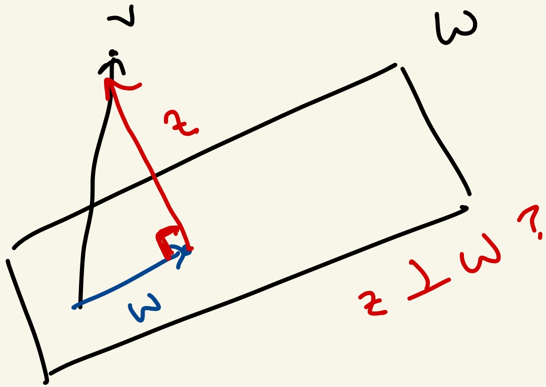
Reminder: Exam 2 11/13 Friday!

- same policies
- study guide + practice on canvas (solutions Wednesday?)

Thm (Orthogonal Decomposition) Let $W \subseteq V$ which is finite dimensional, w/ inner product $\langle -,- \rangle$. Then for all $v \in V$, $v = w + z$ where $w \in W$ and $z \in W^\perp$.

Moreover this sum is unique. $(w \perp z)$



<u>Pf</u>

$$w = \mathrm{proj}_W v$$
$$z = v - \mathrm{proj}_W v.$$

guess?

We need that $w \in W$ and $z \in W^\perp$ and $w + z = v$.

- $w + z = \mathrm{proj}_W v + (v - \mathrm{proj}_W v) = v.$
- $\mathrm{proj}_W v = w \in W$ by definition

- $v - \text{proj}_W v \perp \text{proj}_W v$ but this doesn't mean that $v - \text{proj}_W v \in W^\perp$. $\text{proj}_W v$ is only 1 vector in W, we need to prove that $v - \text{proj}_W v$ is orthogonal to all of W!

Let $u_1 \dots u_k$ be an orthonormal basis of W. If $\boxed{z = v - \text{proj}_W v \perp u_i}$, then $z \perp a_1 u_1 + \dots + a_k u_k$

$z \in W^\perp$.

$$\underbrace{\langle z, u_i \rangle} = \langle v - \text{proj}_W v, u_i \rangle$$

$$= \langle v - c_1 u_1 - c_2 u_2 - \dots - c_k u_k, u_i \rangle$$

where $\underbrace{c_j = \langle v, u_j \rangle}$

$$= \langle v, u_i \rangle - c_1 \overset{0}{\cancel{\langle u_1, u_i \rangle}} - c_2 \overset{0}{\cancel{\langle u_2, u_i \rangle}}$$

$$- \underset{c_i \langle u_i, u_i \rangle}{\dots} - c_k \overset{0}{\cancel{\langle u_k, u_i \rangle}}$$

$$= \langle v, u_i \rangle - c_i \langle u_i, u_i \rangle \rightarrow 1$$

u is a unit vector

$$= \langle v, u_i \rangle - \langle v, u_i \rangle \, 1$$

$$= \underline{0}.$$

$$z \perp u_i \implies z \in W^\perp.$$

So $w = \text{proj}_W v$ $z = v - \text{proj}_W v$ is the orthogonal decomp!

Why is this unique?

Assume $v = w + z = \tilde{w} + \tilde{z}$, v has 2

orthogonal decomps.

Well, $w + z = \tilde{w} + \tilde{z}$

$\implies$ $w - \tilde{w} = \tilde{z} - z$.

In fact, $w, \tilde{w} \in W \implies w - \tilde{w} \in W$.

$$z, \tilde{z} \in W^{\perp} \implies \tilde{z} - z \in W^{\perp}.$$

$$\left(\begin{array}{l} w - \tilde{w} \in W \\ \tilde{z} - z \in W^{\perp} \end{array} \right. \implies \underbrace{\vec{x} = w - \tilde{w} = \tilde{z} - z}$$

$$\vec{x} \in W \cap W^{\perp}.$$

Remember, we showed that $W \cap W^{\perp} = \{0\}$.

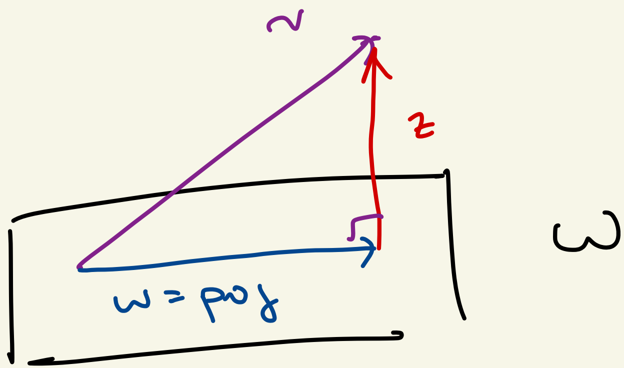
$$\implies \vec{x} = \vec{0}.$$

$$w - \tilde{w} = 0 \implies w = \tilde{w} \, !$$

$$\tilde{z} - z = 0 \implies z = \tilde{z} \, !$$

$v = w + z = \tilde{w} + \tilde{z}$ were

the same all along. $\square$

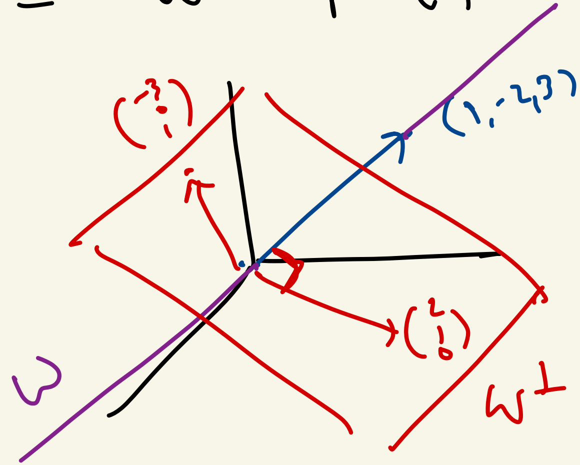


is unique!

So we have a bunch of results about $W, W^{\perp}$.

How do you actually find $W^{\perp}$?

Ex $W = \text{span}(1,-2,3)$ w/ dot product. What is $W^\perp$?



If $z \in W^\perp$, $z \perp (1,-2,3)$.

$z = (z_1, z_2, z_3) \cdot (1,-2,3) = 0.$

$* \quad z_1 - 2z_2 + 3z_3 = 0$

RREF $\longrightarrow$

$$\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0 \qquad z_1 = 2z_2 - 3z_3.$$

free free

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2z_2 - 3z_3 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} z_2 + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} z_3 .$$

$$W^{\perp} = \mathrm{Span}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

Ex $\quad \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right)^{\perp} = ?$ $\quad$ if $\vec{z} = (x, y, z, w) \in W^{\perp}$

$$(1,0,2,3) \cdot z = 0$$
$$(1,-1,-1,1) \cdot z = 0$$
$$\Longrightarrow \quad \begin{aligned} x + 2z + 3w &= 0 \\ x - y - z + w &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

defining eq'n of $W^{\perp}$

rows!!

Use RREF to calculate a basis of $W^\perp$!

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & -1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -\frac{3}{2} & -1 \end{pmatrix}$$

free free

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z - 3w \\ \frac{3}{2}z + w \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} w$$

So $W^\perp = \operatorname{span}\left(\begin{pmatrix} -2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right).$

Thm Let A be a $m \times n$ matrix. Then

$$\ker(A) = \operatorname{coimg}(A)^{\perp} \quad \text{and}$$

$$\operatorname{coker}(A) = \operatorname{Img}(A)^{\perp}.$$

(w/r/t dot product!)

End of exam material!

Pf $x \in \operatorname{coimg}(A)^{\perp}$

$\operatorname{coimg}(A) = \text{span of rows of } A$

$= \operatorname{span}(\vec{r}_1, \ldots, \vec{r}_m)$?

If we start w/ x which is in $\operatorname{coimg}(A)^{\perp}$

* what we were just solving!

$x \in \operatorname{coimg}(A)^{\perp} \Longleftrightarrow x \perp \vec{r}_i$ for all rows $\vec{r}_i$ of A

Remember
$\operatorname{coimg}(A) = \operatorname{span}(\text{rows})$

$$\iff \vec{x} \cdot \vec{r}_i = 0 \quad \forall i \leftarrow \text{for all } i$$

$$\iff \begin{pmatrix} - & r_i & - \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad \forall i$$

$$\iff \underline{A\vec{x}} = \begin{pmatrix} - & r_1 & - \\ & \vdots & \\ - & r_m & - \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{0}$$

$$\iff \underline{x \in \ker(A)}.$$

$$\operatorname{coker}(A) = \ker(A^T) = \operatorname{coimg}(A^T)^\perp = \operatorname{img}(A)^\perp \qquad \square$$

$$\mathrm{Span}\left(\begin{pmatrix}1\\0\\2\\3\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right)^{\perp} = ?$$

$$\mathrm{Colmg}\begin{pmatrix}1 & 0 & 2 & 3\\ 1 & -1 & -1 & 1\end{pmatrix}^{\perp} = \text{solutions to } \begin{pmatrix}1 & 0 & 2 & 3\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$= \ker\begin{pmatrix}1 & 0 & 2 & 3\\ 1 & -1 & -1 & 1\end{pmatrix} \; !$$

Kernel of a matrix A

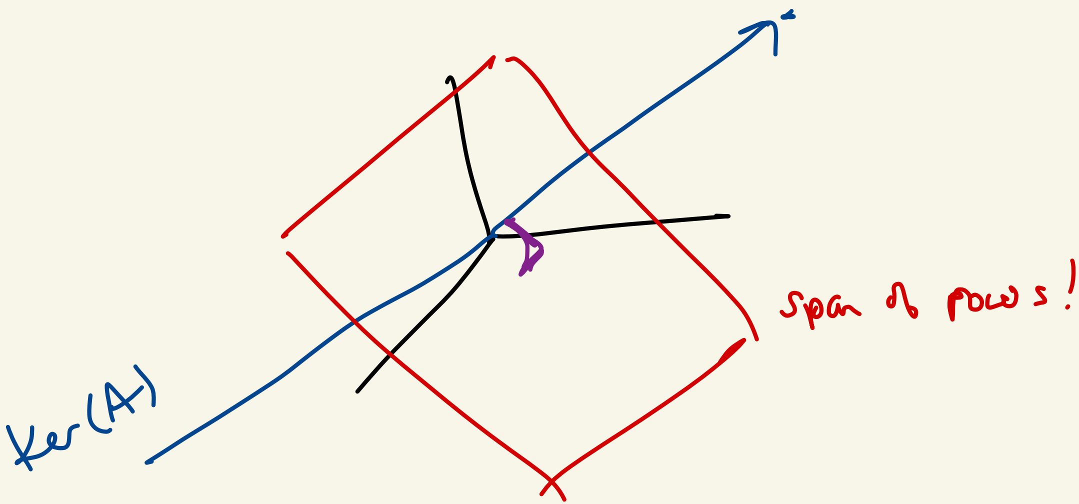
= all solutions to $A x = \vec{0}$.

(= all linear relationships between the columns of A)

img(A) = span of all the columns of A

dim ker + dim img = # of columns

$\mathrm{coimg}(A)^{\perp} = \left(\text{span of rows of } A\right)^{\perp} = \ker(A)$!!



If we know that $\operatorname{coimg}(A)^{\perp} = \ker(A)$,

$\ker(A)^{\perp} = \operatorname{coimg}(A)$? Yes!

cancel

$$\ker(A)^{\perp} = \left(\operatorname{coimg}(A)^{\perp}\right)^{\perp} = \operatorname{coimg}(A).$$

$$(W^{\perp})^{\perp} = W \quad \text{(general principle)}$$

Next time...

$$\text{Proj}_W v = \underbrace{\langle v, u_1 \rangle}_{a_1} u_1 + \underbrace{\langle v, u_2 \rangle}_{a_2} u_2 + \dots + \underbrace{\langle v, u_k \rangle}_{a_k} u_k$$

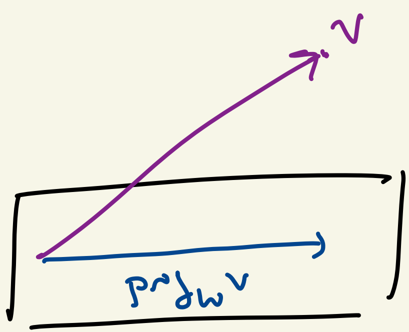
given!

$\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ is an * orthonormal basis of W. *

$$W = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right)$$

1) Find $u_1 \dots u_k$

2) Compute $\text{proj}_W v$ is a vector!



W

the "shadow" of v onto W.

1) $\vec{v}_1 \ldots \vec{v}_k$ orthogonal $\quad a_i = \dfrac{\langle v, v_i \rangle}{\|v_i\|^2}$ $*$

We proved that all inner products look like $\langle \vec{x}, \vec{y} \rangle = x^T K y$ where K is positive definite

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2 = (x_1 \;\; x_2)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

positive definite

$\langle \vec{x}, \vec{y} \rangle = x_1^2 + y_1^2 + x_2^2 + y_2^2$ not an inner product!

bad!

<u>bilinearity</u>.

<u>Ex</u> $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2$ HW

$= (x_1 \ x_2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ not positive!

not positive definite.

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has kernel $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so if $x = (1, -1)$ then this will make it not positive

$$\langle (1,-1),\ (1,-1) \rangle = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.$$

not positive