HW 10 due tonight!

HW 11 due 12/16 (short, 3 problems?)

- I'll keep the gradescope open until Friday

Final 12/21 1:30 - 3:30 *

Office Hours: Next Tuesday 12/15 2:00 - 4:00 pm *

Review Thursday 12/17 12:00 - 3:00 pm

Friday 12/18 + Appt

come w/ questions!

Final Review - posted some yesterday, more to come!
(slightly harder than exam problems)

60% new material 40% cumulative material

8 - 10 questions

Jordan Form – generalized version of diagonalization

Jordan block $\lambda = 0$ need 2 1's

" 3-1

- 1's live on the superdiagonal
- diagonal above the diagonal

__Recall__ __Ex__ $A = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$

A has $\lambda = 0$ w/ alg mult $= \boxed{3}$

$V_0 = \ker(A - 0I) = \text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ geom mult $= \boxed{1}$

$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the only independent eigenvector

__NO__ diagonalization

Turns out all non-diagonalizable matrices are similar to this example.

$A \sim S^{-1}AS = B$

Thm let A be an $n \times n$ matrix. (real entries or complex entries)

Then $A = SJS^{-1}$ where S is a matrix of generalized eigenvectors (Jordan basis)

and J is the Jordan form of A,

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 \\ & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\lambda_k} \end{pmatrix} = \begin{pmatrix} J_{\lambda_1, n} & & \\ & \ddots & \\ & & J_{\lambda, n} \end{pmatrix}$$

Jordan blocks

Ex Suppose that A has eigenvalues $\lambda = 2,2$ $\lambda = -1$

alg mult(2) = 2 geom mult $\boxed{=1}$ I'm lacking 1 eigenvector

alg mult(−1) = 1 geom mult = 1

From this information, we can make the Jordan form

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1^{*} \\ 0 & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{-1} \end{pmatrix}$$

These are the Jordan blocks

Ex Suppose that A $\lambda = 3,3,3,3$ $\lambda = 2,2$

We lack 2 eigenvectors ×

alg mult of 3 = 4 geom mult = 2

alg mult 2 = 2 geom mult = 2 also ✓

There's two possibilities for Jordan form J

$$\begin{pmatrix} \boxed{\begin{matrix} 3 & 1 & \\ & 3 & 1 \\ & & 3 \end{matrix}} & & & \\ & \boxed{3} & & \\ & & \boxed{2} & \\ & & & \boxed{2} \end{pmatrix} \qquad \begin{pmatrix} \boxed{\begin{matrix} 3 & 1 \\ & 3 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 3 & 1 \\ & 3 \end{matrix}} & & \\ & & \boxed{2} & \\ & & & \boxed{2} \end{pmatrix}$$

These are different!!

The positions of the 1's matter.

what are generalized eigenvectors?

Suppose we need to find some missing eigenvectors.

Start w/ an actual eigenvector $\vec{w}_1$, $A w_1 = \lambda_1 w_1$

From w_1, we have to make a Jordan chain

w_1 , w_2 , ... , w_k

actual eigenvector

generalized eigenvectors!

1 generalized eigenvector for each missing eigenvector

Recursive formula: $A\vec{w}_2 = \lambda \vec{w}_2 + \vec{w}_1$

w_2 is a sol'n to

$$(A - \lambda I)\vec{w}_2 = \boxed{\vec{w}_1}$$

step 1

$$- \ (A - \lambda I) w_3 = w_2$$

step 2

$\vdots$

$$- \ (A - \lambda I) w_k = \boxed{w_{k-1}}$$

step k-1

$A - \lambda I$ not invertible

row reduction required

Ex Find the Jordan form and change of basis matrix for $A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$.

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(5-\lambda) + 1 = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

$\underline{\lambda = 4, 4}$

alg mult = 2

geom mult = 1

We are lacking 1 eigenvector!

We need 1 generalized eigenvector.

$V_{\lambda=4}$ = subspace of all eigenvectors for $\lambda = 4$ = $\ker(A - 4I)$

$$= \ker\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$(A - \lambda I) w_1 = 0$

$(A - 4I)\boxed{w_2} = w_1$

We want to solve for w_2

w_1, w_2 is a Jordan chain of length 2

Let's pick $\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ actual eigenvector

$$(A - 4I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

not invertible!

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right) \longrightarrow$$

$$x = y - 1$$

$$w_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y-1 \\ y \end{pmatrix} =$$

$$\vec{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$\vec{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ there's a set of possibilities for w_2.

$y = 0$

Turns out you can't pick $y=0$ everytime.

$\vec{w}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\vec{w}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ generalized eigenvector! Since we are at the end of the chain we can pick $y=0$.

generalized eigenvector!

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$$

gen

needs a 1

Ex $A = \begin{pmatrix} 7 & 1 & 1 \\ 4 & 4 & 4 \\ 1 & -2 & 7 \end{pmatrix}$

This matrix is not diagonalizable!

$\det(A - \lambda I) = 0$

$-(\lambda - 6)^3 = 0 \rightsquigarrow \lambda = 6, 6, 6$

alg mult $= 3$

$V_6 = \ker(A - 6I) = \ker\begin{pmatrix} 1 & 1 & 1 \\ 4 & -2 & 4 \\ 1 & -2 & 1 \end{pmatrix} = \mathrm{span}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

geom mult $= 1$

We need 2 generalized eigenvectors!

$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, w_2, w_3$ is the Jordan chain

① - $(A - 6I) w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

- $(A - 6I) w_3 = w_2$

not invertible

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & -2 & 4 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 4 & -2 & 4 & 0 \\ 1 & -2 & 1 & 1 \end{array}\right)$$

$z = 0$

$$w_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cancel{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} z + \underbrace{\begin{pmatrix} -1/3 \\ -2/3 \\ 0 \end{pmatrix}}$$

parameter

Pick $z = 0$

$$w_2 = \begin{pmatrix} -1/3 \\ -2/3 \\ 0 \end{pmatrix}$$

In general picking $z = 0$ doesn't work, but it does in this case!

Hold off until next time.

2

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & -2 & 4 \\ 1 & -2 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{w_3} = \underbrace{\begin{pmatrix} -1/3 \\ -2/3 \\ 0 \end{pmatrix}}_{w_2}$$

$\longrightarrow$ solve using rref

$$w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} z + \begin{pmatrix} -2/9 \\ -1/9 \\ 0 \end{pmatrix}$$

Now, we are at the end of the chain so we don't have to worry about the next step, every choice for z is fine.

$$z = 0$$

$$w_3 = \begin{pmatrix} -2/9 \\ -1/9 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1/3 \\ -2/3 \\ 0 \end{pmatrix} \qquad w_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & -1/3 & -2/9 \\ 0 & -2/3 & -1/9 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1^{*} & \\ & 6 & 1^{*} \\ & & 6 \end{pmatrix} \begin{pmatrix} \cdots \end{pmatrix}^{-1}$$

generalized