Last time ... Jordan form

Jordan chain $w_1 \dots w_k$ series of generalized eigenvectors

$$A w_1 = \lambda w_1$$

$$(A - \lambda I) w_2 = \boxed{w_1}$$ given eigenvector

$$\vdots$$

$$(A - \lambda I) w_k = w_{k-1}$$

$A = S J S^{-1}$ where S is the Jordan basis

and J is an almost diagonal matrix of eigenvalues, but w/ some 1's above the diagonal everytime you have a generalized eigenvector.

$$A w_2 = \lambda w_2 + w_1 = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} 1 \\ \boxed{\lambda} \end{pmatrix}$$

1 above it

eigenvalue

Usual example :

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & -3 \\ -1 & 1 & -2 \end{pmatrix}$$

Step ① Compute eigenvalues and eigenspaces

$$\det(A - \lambda I) = 0 \implies \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ -1 & 2-\lambda & -3 \\ -1 & 1 & -2-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \lambda^3 = 0$$

$$\lambda^2(1-\lambda) = 0$$

$\lambda = 0, 0$ $\boxed{\text{alg mult} = 2}$

$\lambda = 1$ $\boxed{\text{alg mult} = 1}$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & -3 \\ -1 & 1 & -2 \end{pmatrix}$$

$$V_1 = \ker(A - 1I) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & -3 \\ -1 & 1 & -3 \end{pmatrix}$$

$$= \operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\boxed{\text{geom mult} = 1}$

No generalized eigenvectors for $\lambda = 1$

$$V_{\lambda=0} = \ker(A - 0I) = \ker(A)$$

$$= \operatorname{span}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\boxed{\text{geom mult} = 1}$

- So need 1 generalized eigenvector
- We need a Jordan chain of length 2

$$w_1 = \begin{pmatrix}-1\\1\\1\end{pmatrix} \qquad w_2 \quad \text{s.t.} \quad (A - 0I)w_2 = w_1 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

never invertible!

$$\begin{pmatrix}1 & 0 & 1\\-1 & 2 & -3\\-1 & 1 & -2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

z free

$$\left(\begin{array}{ccc|c}1 & 0 & 1 & -1\\-1 & 2 & -3 & 1\\-1 & 1 & -2 & 1\end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c}\boxed{1} & 0 & 1 & -1\\0 & \boxed{1} & -1 & 0\\0 & 0 & 0 & 0\end{array}\right)$$

$$\begin{aligned} x &= -z-1 \\ y &= z \\ & z \end{aligned}$$

$$w_2 = \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-z-1\\z\\z\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}z + \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

w_2 is the end of the chain so any choice of z will work! Pick $z = 0$

$$\longrightarrow \quad w_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Jordan basis $\quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = \underline{1} \qquad \lambda = 0 \qquad \lambda = 0$

generalized

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & -3 \\ -1 & 1 & -2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 1 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1}}_{S^{-1}}$$

① Put λ on the diagonal

Ex

$$A = \begin{pmatrix} \underline{2} & -1 & 1 & 2 \\ 0 & \underline{2} & 0 & 1 \\ 0 & 0 & \underline{2} & -1 \\ 0 & 0 & 0 & \underline{2} \end{pmatrix}$$

Diagonals of an upper Δ matrix are the eigenvalues!

$$\det(A - \lambda I) = 0$$

$$\lambda = 2, 2, 2, 2 \qquad \text{alg mult of } 4$$

$$\ker(A - 2I) = \ker \begin{pmatrix} 0 & -1 & 1 & 2 \\ & 0 & 0 & 1 \\ & & 0 & -1 \\ & & & 0 \end{pmatrix}$$

$$= \text{span}\ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\#\text{ of ind eigenvectors } = \dim(V_4) = 2} \qquad \text{geom mult } 2$$

We need 2 generalized eigenvectors

$$A = \begin{pmatrix} \underline{2} & -1 & 1 & 2 \\ 0 & \underline{2} & 0 & 1 \\ 0 & 0 & \underline{2} & -1 \\ 0 & 0 & 0 & \underline{2} \end{pmatrix}$$

Unfortunately, guess and check.

Turns out $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ has no chains

$$(A - 2I) w_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

has no solutions!

- ✗ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, w_2 \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, u_2$

doesn't exist

- ✓ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, w_2, w_3 \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

- ✗ $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, u_2, u_3$

$$\begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix} \text{ ✗} \quad \text{vs.} \quad \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{pmatrix} \text{ ✓}$$

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_2, \quad w_3$$

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$ because we can set $x = z = 0$.

Doesn't work

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} w_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

inconsistant!

no solution

What x, z work?

w_2 has to have 2 properties (satisfy 2 eq'ns)

- $(A - \lambda I) w_2 = w_1$

  $\uparrow \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ ✓

- $(A - \lambda I) w_3 = w_2$

  $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ X

$$w_2 \in \left\{ \begin{array}{l} \text{Solution set} \\ \text{of } (A - \lambda I) v_2 = w_1 \end{array} \right\} \cap \boxed{\operatorname{img}(A - \lambda I)}$$

and

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$w_2 = \cancel{c_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}} + \underbrace{c_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}} + c_4 \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= (c_3 - c_2) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= c_1'\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_4\begin{pmatrix}2\\1\\-1\\0\end{pmatrix}$$ Solve for c_1' and c_4

Since $(A - \lambda I)w_2 = v_1$

$$\begin{pmatrix}0&-1&1&2\\0&0&0&1\\0&0&0&-1\\0&0&0&0\end{pmatrix}\left(c_1'\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + c_4\begin{pmatrix}2\\1\\-1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\underbrace{c_1\begin{pmatrix}0&-1&1&2\\0&0&0&1\\0&0&0&-1\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{\cancel{c_1\begin{pmatrix}0\\0\\0\\0\end{pmatrix}}} + c_4\begin{pmatrix}0&-1&1&2\\0&0&0&1\\0&0&0&-1\\0&0&0&0\end{pmatrix}\begin{pmatrix}2\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$c_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad c_4 = \frac{-1}{2}$$

c_1 anything

$c_1 = 0$

$$w_2 = -\frac{1}{2}\begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ -1/2 \\ 1/2 \\ 0 \end{pmatrix}}_{\text{generalized eigenvector.}}$$

$$w_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1/2 \end{pmatrix}$$

this order matters

$$\begin{pmatrix} 2 & -1 & 1 & 2 \\ & 2 & 0 & 1 \\ & & 2 & -1 \\ & & & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1/2 & 0 & 1 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & -1/2 & 0 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}^{-1}}_{S^{-1}}$$

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generalized

What is Jordan form good for?

Suppose we have a linear system of differential eq'ns.

$x(t), y(t) \rightsquigarrow \vec{v}(t) = (x(t), y(t))$

$$\frac{dx}{dt} = 2x(t) + 3y(t)$$

$$\frac{dy}{dt} = -x(t) + y(t)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2x(t) + 3y(t) \\ -x(t) + y(t) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\vec{v}(t)' = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \vec{v}(t)$$

$$\frac{d\vec{v}}{dt} = A\vec{v}$$ where $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}$

Normally . . .

$$d\vec{v} = A\vec{v}\, dt$$

$$\frac{1}{\vec{v}}\, d\vec{v} = A\, dt$$

$$\int \frac{1}{\vec{v}}\, d\vec{v} = \int A\, dt$$

None of this is actual math, heuristic

$$e^{\ln(\vec{v})} = e^{At + \vec{c}}$$

$$\underbrace{\vec{v}(t)} = e^{At} e^{\vec{c}} = \underbrace{e^{At} \vec{v}(0)}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \boxed{e^{At}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \qquad e^{\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} t}$$

what does this mean ??

$$\text{"} e^{At} \text{"}$$

Key idea: There's a way to make sense of this heuristic.

Thm / Def: let A be a square $n \times n$ matrix.

Def $$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}A^n \quad \text{where } A^0 = I$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$$

Thm This converges?

e^A can be computed w/ Jordan form.