HW 11 due tonight!

- I'm going to grade it Fri @ 3pm

  Feel free to upload it before then.

Last time ...

e^{At} sol'n to

$\vec{v}'(t) = A v(t)$

$\Longrightarrow \vec{v}(t) = e^{At} v(0)$

$$e^A = I + A + \frac{1}{2} A^2 + \frac{1}{3!} A^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

what is e^A ?

But what is it explicitly? How do you calculate it?

<u>Prop</u>

- e^A respects change of basis.

  If $A = SJS^{-1}$ then

  $$e^A = S e^J S^{-1}$$

- If $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$

  then $e^D = \begin{pmatrix} e^{d_1} & & & 0 \\ & e^{d_2} & & \\ & & \ddots & \\ 0 & & & e^{d_n} \end{pmatrix}$

$$\boxed{e^{x+y} = e^x e^y}$$

- If $AB = BA$ then

  $$e^B e^A = e^{A+B} = e^A e^B$$

Quick Pf

- $$e^A = \sum \frac{1}{n!} A^n = \sum \frac{1}{n!} (SJS^{-1})^n$$

$$(SJS^{-1})^3 = SJ\cancel{S^{-1}S}J\cancel{S^{-1}S}JS^{-1} = SJ^3S^{-1}$$

$$= \sum \frac{1}{n!} SJ^nS^{-1} = S\left(\sum \frac{1}{n!} J^n\right)S^{-1}$$

$$= Se^JS^{-1}$$

- $$e^D = \sum \frac{1}{n!} \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_k \end{pmatrix}^n = \sum \frac{1}{n!} \begin{pmatrix} d_1^n & & \\ & \ddots & \\ & & d_k^n \end{pmatrix}$$

$$= \begin{pmatrix} \sum \frac{1}{n!} d_1^n & & \\ & \ddots & \\ & & \sum \frac{1}{n!} d_k^n \end{pmatrix} = \begin{pmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_k} \end{pmatrix}$$

• $\underline{\text{If} \quad AB = BA}$ $\qquad e^{A+B} = I + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \dots$

$$\underbrace{\frac{1}{2}(A+B)^2} = \frac{1}{2}\left(A^2 + AB + BA + B^2\right)$$

if $AB \neq BA$ then this the best you can simplify

$$= \frac{1}{2}\left(A^2 + 2AB + B^2\right)$$

$$e^{A+B} = I + (A+B) + \frac{1}{2}(A^2 + 2AB + B^2) + \frac{1}{3!}\left(A^3 + 3A^2B + 3AB^2 + B^3\right) + \dots$$

really depends on $AB = BA$

$$= \left(I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots\right)\left(I + B + \frac{1}{2}B^2 + \frac{1}{3!}B^3 + \dots\right)$$

$$= e^A e^B$$

$\square$

Ex Compute $e^{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}}$.

Step 1 Find the Jordan decomposition of $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$.

$$e^{SJS^{-1}} = S e^{J} S^{-1}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

generalized eigenvector

$$e^{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \underbrace{e^{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}}_{\text{simpler to compute}} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

Step 2

$$e^{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}} = e^{\overset{\Lambda}{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} + \overset{N}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}}$$

N ← matrix of all the 1's

$$e^{A+B} = e^{A} e^{B}$$

$$AB = BA$$

In fact $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

turns out Δ, N always commute when $J = \Delta + N$

$$e^{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}} = e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

Step 3

$$e^{D} = \begin{pmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_n} \end{pmatrix}$$

$$e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(nilpotent matrix)

$N^k = 0$

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\underbrace{\cancel{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2}}_{0} + \frac{1}{3!}\underbrace{\cancel{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^3}}_{0} + \underbrace{\cancel{\cdots}}_{0} \; ---$$

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Step 4 Put it all together!

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e^{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} e^{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

Step 2

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} e^{\boxed{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{1}{-1} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^2 & e^2 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$e^{\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}} = \boxed{\begin{pmatrix} 0 & -e^2 \\ e^2 & 2e^2 \end{pmatrix}}$$

Idea: These steps work for all matrices.

Ex $e^{\begin{pmatrix} 7 & 8 \\ -4 & -5 \end{pmatrix}}$

Step 1 $\begin{pmatrix} 7 & 8 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$

$$e^{\begin{pmatrix} 7 & 8 \\ -4 & -5 \end{pmatrix}} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} e^{\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

Step 2, Step 3

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e^{3} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$e^{\begin{pmatrix} 7 & 8 \\ -4 & -5 \end{pmatrix}} = \begin{pmatrix} 2e^{3} - e^{-1} & 2e^{3} - 2e^{-1} \\ -e^{3} + e^{-1} & -e^{3} + 2e^{-1} \end{pmatrix}$$

Ex

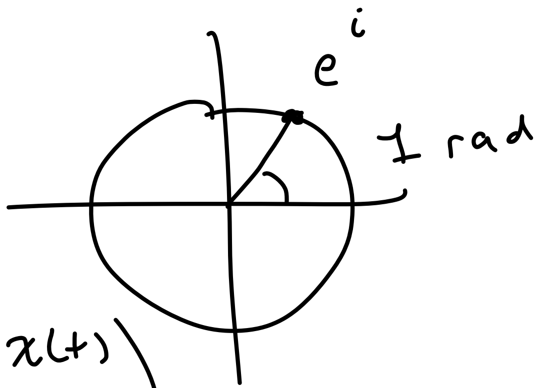
$$e^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}} = \begin{pmatrix} e^0 & 0 & 0 \\ 0 & \boxed{e^{-i}} & 0 \\ 0 & 0 & e^i \end{pmatrix}$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$x = \theta$ angle

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

$x = 1$, $e^i = \cos(1) + i\sin(1)$



$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x$$

$$\Rightarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

__Ex__ $e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$

$$\det\begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$V_i \subseteq \mathbb{C}^2$$

$$V_i = \ker(A - iI) = \ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\xrightarrow{ir_1 + r_2} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \qquad \text{RREF}$$

$x = iy \quad y$ free

$$V_i = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} iy \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} y \qquad v = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$V_{-i} = \operatorname{span} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1}$$

$$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} e^{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}} \frac{1}{\det} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

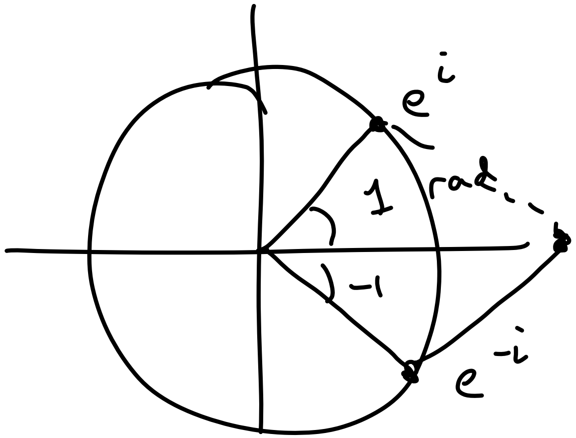
$$= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{pmatrix} \frac{1}{2i} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i} & 0 \\ 0 & e^{-i} \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$= \frac{1}{2i}\begin{pmatrix} ie^{i} & -ie^{-i} \\ e^{i} & e^{-i} \end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$= \frac{1}{2i}\underbrace{\begin{pmatrix} ie^{i} + ie^{-i} & -e^{i} + e^{-i} \\ e^{i} - e^{-i} & ie^{i} + ie^{-i} \end{pmatrix}}_{e^{i},\ e^{-i}}$$

you can stop on HW 10.4.2a



$$e^{i} + e^{-i} = 2\cos(1)$$

$$e^{i} = \cos(1) + i\sin(1)$$

$$e^{-i} = \cos(-1) + i\sin(-1) = \cos(1) - i\sin(1)$$

So $e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ will be real if you keep simplifying

Ex

$$e^{\begin{pmatrix} 1 & 5 & -3 \\ -2 & -4 & 2 \\ -3 & -1 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 5 & -3 \\ -2 & -4 & 2 \\ -3 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & \\ & -2 & \\ & & 2 \end{pmatrix} \begin{pmatrix} \cdots \end{pmatrix}^{-1}$$

generalized eigenvector

$$= \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 0 & 2 \end{pmatrix} e^{\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 0 & 2 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 0 & 2 \end{pmatrix} e^{\begin{pmatrix} -2 & & \\ & -2 & \\ & & 2 \end{pmatrix}} e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{1}{2} & -1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$S \begin{pmatrix} e^{-2} & & \\ & e^{-2} & \\ & & e^{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1} = \cdots$$

$$e^N = I + N$$

HW 11 due tonight (Fri 3pm)

Office Hrs tomorrow 12/17 usual time 12 - 3

Office Hrs Friday 12/18 12 - 3

Final Monday 12/21 1:30 - 3:30

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x + z = 0$

$y = -1$

z free

(1 - 1½

$Ax = b$

$$w_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ -1 \\ z \end{pmatrix} = \cancel{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} z}^{\,0} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

ker(A) + particular solution

$$w_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$